Would Letting People Vote for Several Candidates Yield Policy Moderation?

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The present paper investigates whether letting people cast a vote for several candidates instead of only one would yield more moderate policies. It does so in a setting that allows for endogenous candidate entry, strategic voting and policy-motivated candidates. Two broad classes of voting rules are considered. In the first one, each voter is given several votes to cast, with each vote weighed equally. In the second one, each voter is asked to submit a ranking of the candidates. In both cases, three types of ballots are considered: completely-filled ballots, truncated ballots and cumulated ballots. The paper identifies conditions under which a voting procedure yields policy moderation. It also shows that if any of those conditions does not hold, then letting people vote for several candidates may have the opposite effect, i.e., yield policy extremism instead of policy moderation.

Key Words: Voting rules; Electoral competition; Policy moderation; Citizen-candidate model.

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1. INTRODUCTION

Many countries—like the U.S. or the U.K.—elect their policy-makers by means of Plurality Voting (hereafter PV), the electoral system that allows each voter to vote for one (and only one) candidate and where the candidate who gets the most votes wins the election. Several scholars as well as people’s belief hold that replacing this electoral system with one that would allow citizens to cast several votes would yield policy moderation. The argument is based on the observation that the wasting-the-vote phenomenon—i.e., the voters’ tendency to ignore candidates who are perceived as sure losers because of the fear of wasting one’s vote on that candidate1—creates a barrier to entry by new candidates. Indeed, in countries where elections are held under PV, there is a tendency on the part of voters to perceive new candidates as unlikely winners and, therefore, fear wasting their vote on them. As a result, voters tend to ignore those candidates, even though some of them may be preferred by a majority. The thinking then goes that if citizens were allowed to cast several votes, they would no longer fear wasting one of their votes on a new candidate. This would lower the barriers to entry by new candidates, especially moderate ones whom, it is argued, a large fraction of the electorate

1 For example, in the 2004 U.S. Presidential election, some of Nader’s supporters chose to vote for Kerry or Bush instead of Nader because they anticipated Nader had no chance of winning the election.
regard as acceptable potential decision-makers. Hence the policy moderation since, if elected, those candidates would implement centrist policies.

The present work investigates whether letting people vote for several candidates instead of only one would indeed lead to the adoption of more moderate policies. For that purpose we consider a community that elects a decision-maker to choose a policy, such as a tax rate or the level of public spending. We adopt the citizen-candidate approach of political competition due to Osborne and Sivinski (1996) and Besley and Coate (1997). Under this approach the political process is modeled as follows. First, policy-motivated citizens decide whether to stand for election at a cost. Then, an election is held where citizens choose strategically for whom to vote among the self-declared candidates. Finally, the winner of the election assumes office and chooses the policy to implement. This setting thus allows for strategic voting, strategic candidacy and policy-motivated candidates, all three features that have received empirical as well as theoretical support.2

Before going further we need to specify the voting rules we shall consider.3 The most widely used and studied are the scoring rules where a voter’s ballot takes the form of a vector of points that are cast for the different candidates. All the points a candidate gets are then added up, and the candidate with the most points is declared the winner. In this paper, we shall consider two broad classes of scoring rules. The first consists of all the scoring rules where each vote is weighed equally. We shall call them Multiple Vote scoring rules (hereafter MVs). An example of MV is Dual Voting where a citizen can vote for two candidates, effectively giving one point to each. Another example is Negative Voting where a citizen can vote for all but one of the candidates, effectively giving one point to all the candidates for whom he votes and zero to the one for whom he does not vote. The second class of scoring rules—that we shall call Ordinal Vote scoring rules (hereafter OVs)—consists of all the scoring rules where each voter is asked to rank-order the candidates and where the number of points a candidate gets increases with his position in the voter’s ranking. An example of OV is the Borda Count where the candidate who is ranked last on a voter’s ballot gets no point, the one who is ranked one place above gets one point, the one above two points, and so on.

Besides, for each voting rule we need to specify which ballots are admissible (i.e., valid). In this paper, we shall consider three different types. The first one is the completely-filled ballot, where a voter has to cast all his votes and to cast each of them for a different candidate. For example, if the election is held under Dual Voting, then a ballot is admissible if it has on it the names of exactly two candidates. If instead, the election is held under the Borda Count, a ballot is then admissible if it consists of a ranking of all the candidates. The second type of ballot we shall consider is the truncated ballot, where a voter is given the option not to

2Empirical evidence suggest indeed that a decision-maker’s own policy preferences play a critical role in his policy decisions (see, for example, Levitt 1996). Strategic voting has received both empirical and experimental support (see, for example, Cox 1997 and Forsythe et al. 1996). Furthermore, the relation between the number of political parties and the voting procedure suggests that candidacy is not exogenous indeed. It is worth noting that accounting for those three features can be motivated by theoretical arguments as well. Indeed, any voting rule is subject to strategic voting whenever there are at least three alternatives among which to choose (Gibbard 1973 and Satterthwaite 1975). Moreover, any non-dictatorial and unanimous voting rule is subject to strategic exit and entry by candidates (Dutta et al. 2001). Finally, whether candidates are policy-motivated or not—and thus whether they can commit to implementing a policy if elected—is key for policy moderation (see, for example, Alesina 1988).

3For a brief overview of the most-commonly discussed voting rules, see Farrell (2001) and Mueller (2003).
cast all his votes. For example, if the election is held under Dual Voting, then a voter can choose to cast only one of his two votes. If the election is held under Negative Voting instead, a voter can choose to cast a vote for as many candidates as he wishes.\footnote{Thus, Negative Voting with truncated ballots is equivalent to Approval Voting, a voting rule which had been popularized by Brams and Fishburn (1978, 1983). This voting rule is currently used by several academic and professional associations to elect their officers (see Brams and Fishburn 2005) and its use for political elections has been advocated by several scholars.} If the election is held under the Borda Count, a voter can then report only his top-ranked candidate, or his two top-ranked candidates, and so on. Finally, the third type of ballot we shall consider is the cumulated ballot, where a voter is given the option to cumulate his votes behind fewer candidates. For example, if the election is held under Dual Voting, then a voter can cast one vote for each of two candidates or cast two votes for a single candidate.

The present analysis yields a number of new insights. The first set of results identify conditions under which a voting rule that let citizens vote for several candidates yields policy moderation. In the context of electoral races where all the candidates are serious contenders—i.e., have some chance of being elected the decision-maker—, we find that OVs yield policy moderation. At the same time, a MV yields policy moderation if the following three conditions hold: (1) citizens’ policy preferences are symmetric; (2) each citizen is given ‘enough’ votes to cast; and (3) truncated ballots are admissible. However, if any of those conditions is not satisfied, then moderation need not arise. Instead, extremism may be the result. Indeed, the presence of spoilers—i.e., candidates who run not to win the election but because their presence in the race yields a different electoral outcome—may act as a threat, deterring candidates from entering or exiting the race. Moreover, in an election held under a MV where citizens are given few votes to cast, the wasting-the-vote phenomenon will reappear with multiple candidacies. Furthermore, if vote truncation is not allowed, citizens may have no other choice than either abstaining, or voting for a candidate they dislike. This lowers the barriers to entry for all candidates (moderates as well as extremists), which may then result in a situation where a plethora of candidates are running all over the place. Finally, if policy preferences are not symmetric, some citizens will find it more profitable than others to get their preferred policy adopted. Multiple similar candidacies may then follow, thus reducing others’ chances of becoming the policy-maker which, in turn, dampens their incentives to run for election.

A second set of results looks at the effect of the different ballots on policy moderation. We find that letting citizens vote for several candidates while allowing cumulated ballots yields the same policy outcomes as PV. This is not surprising given that citizens have an incentive to stack all their votes on one candidate. In contrast, the effect of allowing for truncated ballots varies with the voting rule. We have already mentioned that for MVs, moderation is uncertain when truncated ballots are not admissible. And when they are, the same holds true for those MVs with few votes (e.g., Dual Voting). For OVs, it depends on the second-place score in three-way races. More specifically, we distinguish two types of OVs: those with a ‘high’ second-place score, and the others. Using the same terminology as Myerson (1999), we shall call the former worst-punishing and the latter best-rewarding. While vote truncation yields more moderate policy outcomes in the case of best-rewarding OVs, the reverse holds true for worst-punishing OVs—like, for example, the Borda Count. This is because those voting rules differ in their ability to deter and accommodate multiple similar candidacies.
Concerning the comparison between MVs and OVs, we show that with truncated ballots admissible and symmetric policy preferences, the extent of moderation is more substantial under a MV that gives a citizen enough votes to cast—e.g., Approval Voting—than under any OV. However, by the same logic as above, extremism instead of moderation may be the result if preferences are not symmetric or citizens are given too few votes to cast. Moreover, even if those conditions hold, the degree of moderation is more substantial under a worst-punishing OV with only completely-filled ballots admissible—e.g., the Borda Count—than under any MV with or without truncated ballots admissible. This result follows again from the inability of the former ones to deter and accommodate several serious contenders standing on the same platform.

A third set of results concerns the size of the equilibrium set. This is an important question since, as Myerson and Weber (1993) note, a larger multiplicity of equilibria leaves more room for focal manipulation by political actors. We find that the equilibrium set under a MV is a subset of the equilibrium set under PV if vote truncation is allowed, but a superset otherwise. Comparing two OVs, the equilibrium set of the one with the highest second-place score is a subset of the other one. Moreover, the equilibrium set under a worst-punishing OV with only completely-filled ballots admissible is a subset of the equilibrium set with truncated ballots admissible. The reverse holds true for a best-rewarding OV.

A last set of results looks at majority representation. We show that there exists a risk of lopsidedness when the election is held under a MV and only completely-filled ballots are admissible. All the candidates may then be located on the same side of the median citizen’s ideal policy, and only a minority of the electorate is represented in the election. The same can happen in elections held under an OV but only if there are spoilers in the race.

The remainder of the paper is organized as follows. Section 2 reviews the related literature. Section 3 presents the model. Section 4 studies policy moderation in the context of serious races. Section 5 refines the voting behavior and discusses the issue of races with spoilers. Section 6 concludes. All proofs are in the appendix.

2. RELATED LITERATURE

The present paper contributes to a vast literature on the effect of political competition on policy moderation. The first strand of this literature studies PV elections in the Downsian framework. In his seminal contribution, Downs (1957) considers elections where two purely office-motivated candidates compete by choosing a platform along a one-dimensional policy space. Information is complete and perfect. In this context, both candidates adopt the median voter’s ideal policy, and the extent of moderation is maximal.

However, this result is rarely observed in practice. Subsequent contributions have then tried to resolve this dissonance between theory and practice. Two approaches have been followed. The first one keeps the set of candidates exogenously given. In this context, Wittman (1983), Calvert (1985) and Roemer (1997) show that we can get polarization in the Downsian framework if the candidates are both policy-motivated and uncertain about the distribution of voters’ policy preferences. But, as Calvert (1985) notes, the extent of polarization need not be substantial. The latter is no longer true however if the policy making is shared by a legislative and an executive branch of the government (Alesina and Rosenthal 2000) or if the candidates are nominated by political parties (Fauli-Oller et al. 2003). Oth-
ers have shown that polarization can also arise in the Downsian framework if we allow for costly voting and abstention (Feddersen 1992), the presence of interest groups (Baron 1994) or the possibility that one of the contenders is an incumbent (Bernhardt and Ingberman 1985).

The second approach endogenizes the set of candidates. Palfrey (1984), Weber (1992, 1998) and Callander (2005) add the threat of entry to the two-candidate competition of the Downsian model. They then show that—except if entry is very costly (Weber 1998)—the two established contenders have no other choice than to run on polarized platforms if they want to deter entry or limit its impact. At the same time, Feddersen et al. (1990) show that when citizens vote strategically and the whole set of candidates is endogenized—i.e., all potential candidates choose simultaneously whether to stand for election at a cost—, then all the potential candidates who enter the race do so at the median voter’s ideal policy. However, endogenizing the timing of entry brings polarization back (Osborne 1993 and 2000).

More relevant for the present contribution is a second strand of the literature that, within the Downsian framework, compares policy moderation under different electoral systems. Cox (1987) studies single-winner elections with a fixed set of (at least three) candidates, whose only interest is in winning the election. Candidates compete by choosing a platform along a one-dimensional policy space and citizens vote sincerely. He finds that, in the context of elections where all candidates choose to stand on the same platform—equilibria that he calls convergent—, PV is one of the few among the commonly-discussed voting rules in not having moderate outcomes. This contribution contrasts with ours both in terms of results and modeling—we adopt the citizen-candidate approach and allow for strategic voting as well as non-convergent equilibria.

Using the same framework as in his 1987 paper, Cox (1990) considers elections held under MVs and studies how policy moderation depends on the number of decision-makers to elect, the number of votes a citizen can cast and the possibility of vote cumulation or vote truncation. He finds that the extent of policy moderation increases with the number of votes a citizen can cast. The same is true in the present analysis, but only if some conditions hold. In addition, he finds that the admissibility of truncated ballots has a negative effect on policy moderation and that in single-seat two-way races where cumulated ballots are admissible, the candidates stand at the median voter’s ideal policy. Those two results are in sharp contrast with ours.

Myerson and Weber (1993) allow for strategic voting, adopting the pivotal-voter approach whereby a citizen conditions his vote on his anticipation of a close race between the different candidates. They consider two voting rules, PV and Approval Voting. They find that PV imposes few restrictions on the location of serious contenders, while all are running at the median voter’s ideal policy when the election is held under Approval Voting. The difference lies in the wasting-the-vote phenomenon. While we reach the same conclusion about PV, we get that the extent of moderation under Approval Voting depends on several conditions—namely, the symmetry of preferences and the absence of spoilers in the race.5

A more recent strand of the literature—the citizen-candidate approach initiated

5The Downsian framework has also been used to study, for example, the effect of different electoral systems on reducing government corruption (see, for example, Myerson 1993a), on the provision of public goods (see, for example, Lizzeri and Persico 2001 or Milesi-Ferretti et al. 2002) or on the incentives to favor minority interests (see, for example, Myerson 1993b or Lizzeri and Persico 2005). See Myerson (1999) for an interesting review.
by Osborne and Slivinski (1996) and Besley and Coate (1997)—departs from the Downsian framework. The set of candidates is here endogenized by letting the policy-motivated citizens each decide whether to stand for office at a cost, and the citizens vote among the self-declared candidates. The citizen who is elected implements his preferred policy. This approach yields polarized races under PV. This is because entry is costly and candidates have policy preferences, the latter preventing them from committing to adopting a policy different from their ideal one if elected. Previous works in this literature are focused on PV. Notable exceptions are Osborne and Slivinski (1996) who compare policy outcomes under PV and Plurality Runoff, Hamlin and Hjortlund (2000) and Morelli (2004) who study elections under Proportional Representation, and Dellis and Oak (2006) who compare PV and Approval Voting. The present contribution extends this literature by studying policy moderation under the different MVs and OVs.

3. THE MODEL

Consider a community that has to elect a decision-maker to choose a policy, such as a tax rate or the level of public spending. The community is made up of \(N\) citizens, indexed by \(c \in N \equiv \{1, \ldots, N\}\). The set of policy alternatives is \(X = \mathbb{R}\).\(^6\)

Citizens’ preferences over the set of policy alternatives are represented by a utility function \(u^c : X \to \mathbb{R}\), which is assumed to be strictly concave.\(^7\) Let \(x^c = \arg \max_{x \in X} u^c(x)\) denote citizen \(c\)'s ideal policy. We assume throughout the analysis that citizens differ only in their ideal policy—that is, \(u^c(x) \equiv u(x^c - x)\) for all citizen \(c\)—with several citizens sharing the same ideal policy.\(^8\) We normalize \(u^c(x^c)\) to 0 and denote by \(v^c_i \equiv u^c(x_i)\) the utility citizen \(c\) derives from \(x_i\), citizen \(i\)'s ideal policy. Furthermore, let \(m\) denote the median citizen’s ideal policy and \(M\) the number of citizens who share this ideal policy.\(^9\) Note that the median citizen’s ideal policy is the Condorcet winner—that is, the policy that defeats any other in a pairwise contest. Finally, let there be \(N - 2M \geq 6\) citizens on either side of \(m\), and assume \(M \leq \frac{N - 4}{3}\).\(^10\)

There are three stages to the policy-making process. At the first stage, each citizen decides whether to stand for office at a cost. Decisions are made simul-

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\(^6\)Assuming the policy space to be unidimensional is made to facilitate comparisons with the previous literature which, for the most part, has been developed under this assumption. Besides, it is a reasonable approximation for many countries (as noted by Cox 1990) as well as for Congressional voting (as shown by Poole and Rosenthal 1991).

\(^7\)Assuming that concavity is strict is made to eliminate the knife-edge PV equilibria where three serious contenders are running for election, each standing on a different platform. This is not a very restrictive assumption however in the sense that those equilibria are non-generic—they can be supported only if preferences are Euclidean and citizens’ ideal policies are distributed in a very specific way. Besides, those equilibria run counter to empirical as well as experimental evidence (see, for example, Cox 1997 and Forsythe et al. 1996).

\(^8\)The assumption that citizens differ only in their ideal policy is made to simplify the analysis. All but one of our results can be derived without this assumption. Moreover, for the result that makes use of it—namely, Proposition 4—, it is only as a sufficient condition. The assumption that several citizens share the same ideal policy is not key either. It is only made to avoid corner solutions where more citizens than there are would be willing to stand on a platform. Relaxing this assumption would complicate the statements of the results without adding any new insight.

\(^9\)In the remainder of the paper, we shall abuse notation and call the median citizen \(m\).

\(^10\)Assuming there are at least six citizens on either side of the median is made to rule out situations where too few votes are cast. Assuming less than a third of the electorate is at the median is made to rule out situations where the citizens at the median would be able to elect any candidate without the support of others.
taneously. At the second stage, an election is held where each citizen decides for whom to vote among the self-declared candidates. Scores are then added up and the candidate who gets the highest vote total is elected the decision-maker. If several candidates tie for first place, each is elected with an equal probability. At the third stage, the newly elected decision-maker chooses the policy to be adopted. It is assumed that a candidate cannot commit to the policy he will implement if elected. This assumption has received empirical justification (see Lee et al. 2004). We now analyze each of these stages in a reverse order.

**Policy-making stage.** Because this stage is the last one, the elected candidate chooses to implement his ideal policy. In case no one runs for election, the status-quo policy $x_0 \in X$ is assumed to be kept.

**Election stage.** Suppose the election is held under $V$ an arbitrary MV or OV. Let $\mathcal{C} \subseteq \mathcal{N}$ be a non-empty set of candidates. Denote by $A_V(\mathcal{C})$ the set of admissible voting strategies under $V$ and let $\alpha^\ell(\mathcal{C}) \in A_V(\mathcal{C})$ be citizen $\ell$’s voting strategy, where $\alpha^i(\mathcal{C})$ is the score citizen $i$ gives to candidate $\alpha$.$^{11}$ We denote the profile of voting strategies by $\alpha(\mathcal{C}) = (\alpha^1(\mathcal{C}), \ldots, \alpha^N(\mathcal{C}))$. We shall call the *winning set* $W(\mathcal{C}, \alpha)$ the set of candidates with the highest score when $\alpha(\mathcal{C})$ is the voting profile; that is,

$$W(\mathcal{C}, \alpha) \equiv \left\{ i \in \mathcal{C} : \sum_{\ell \in \mathcal{N}} \alpha^\ell_i(\mathcal{C}) \geq \sum_{\ell \in \mathcal{N}} \alpha^\ell_j(\mathcal{C}) \text{ for any } j \in \mathcal{C} \right\}.$$  

Given our tie-breaking assumption, citizen $i$’s probability of becoming the decision-maker is then equal to $\frac{1}{\#W(\mathcal{C}, \alpha)}$ if $i \in W(\mathcal{C}, \alpha)$ and 0 otherwise, where $\#W(\mathcal{C}, \alpha)$ is the number of winning candidates. Voter $\ell$’s expected utility is thus given by

$$V^\ell(\mathcal{C}, \alpha) \equiv \frac{1}{\#W(\mathcal{C}, \alpha)} \sum_{i \in W(\mathcal{C}, \alpha)} v^\ell_i.$$  

We are now ready to define a (pure-strategy) voting equilibrium.

**Definition 1** (Voting Equilibrium). Given a non-empty set of candidates $\mathcal{C}$, a strategy profile $\alpha^* (\mathcal{C})$ is a voting equilibrium if for any citizen $\ell$ we have $\alpha^* (\mathcal{C}) \in A_V(\mathcal{C})$ and

1. $V^\ell(\mathcal{C}, \alpha^*, \alpha^{\ell \neq \ell}) \geq V^\ell(\mathcal{C}, \alpha^\ell, \alpha^{\ell \neq \ell})$ for all $\alpha^\ell (\mathcal{C}) \in A_V(\mathcal{C})$;
2. $\alpha^{\ell *}(\mathcal{C}) = (0, \ldots, 0)$ whenever $v^i = v^j$ for all two candidates $i$ and $j$; and
3. $\alpha^{\ell *}(\mathcal{C})$ is weakly undominated. ||

The first condition says that a citizen chooses a voting strategy that maximizes his expected utility, given others’ voting strategies and his anticipation of the policy each candidate will implement if elected. The second condition states that a citizen who is indifferent between all candidates abstains from voting. Finally, the third condition is a common refinement of the voting behavior. The following lemma—adapted from Dellis (2006)—characterizes the set of weakly undominated voting strategies under an arbitrary MV or OV.

**Lemma 1** (Dellis 2006). Take $\mathcal{C}$ a non-empty set of candidates and suppose that citizen $\ell$ is not indifferent between all the candidates in $\mathcal{C}$. A voting strategy $\alpha^\ell(\mathcal{C}) \in A_V(\mathcal{C})$ is weakly undominated for citizen $\ell$ if, and only if, there is no other voting

$^{11}$For example, under PV, $\alpha^\ell(\mathcal{C}) \in A_V(\mathcal{C})$ if either $\alpha^\ell(\mathcal{C}) = (0, \ldots, 0)$—that is, citizen $\ell$ abstains from voting—or $\alpha^\ell(\mathcal{C}) = 1$ for some candidate $i$ and $\alpha^j(\mathcal{C}) = 0$ for any other candidate $j$—that is, citizen $\ell$ votes for candidate $i$. 

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strategy \( \tilde{\alpha}^\ell (C) \in A_V (C) \) such that for all pairs of candidates \( i \) and \( j \), \( v^\ell_i \geq v^\ell_j \) implies 
\[
\left[ \tilde{\alpha}^\ell_i (C) - \alpha^\ell_i (C) \right] \geq \left[ \tilde{\alpha}^\ell_j (C) - \alpha^\ell_j (C) \right].
\]

Thus, for a citizen’s voting strategy to be weakly undominated, it must be that there does not exist another admissible voting strategy which is such that the ordering of score differences between the two voting strategies corresponds to the citizen’s preference ordering over the set of candidates.\(^{12}\)

**Candidacy stage.** Citizens decide simultaneously whether to stand for election at a utility cost \( \delta > 0 \). We assume all citizens anticipate the same voting equilibrium when making their candidacy decision. Denote by \( e^\ell \in \{0, 1\} \) citizen \( \ell \)'s candidacy decision, with \( e^\ell = 1 \) if he chooses to do so and 0 otherwise. Let \( e = (e^1, \ldots, e^N) \) be the profile of candidacy decisions and define \( C(e) \equiv \{ \ell \in N : e^\ell = 1 \} \) as the set of candidates. Thus, for a given profile of entry decisions \( e \), citizen \( \ell \)'s expected utility is \( U^\ell(C(e), \alpha) \equiv \left[ V^\ell(C(e), \alpha) - \delta e^\ell \right] \) if \( C(e) \) is not empty and \( u^\ell(x_0) \) if it is.

We are now ready to define a candidacy equilibrium.

**Definition 2** (Candidacy Equilibrium). A profile of candidacy strategies \( e^* \) is a candidacy equilibrium if for each citizen \( \ell \), \( e^\ell^* \) is a best response to \( e^{-\ell^*} \) given the profile of voting strategies. \( \Box \)

Throughout the analysis, we shall keep the assumption that a citizen decides to stand for election when he is indifferent between entering the race or not.\(^{13}\)

**Political equilibrium.** A political equilibrium (hereafter an equilibrium) is a Subgame Perfect Nash Equilibrium of this game. It consists of a pair \( (e^*, \alpha^*(\cdot)) \) where: (1) \( \alpha^*(\cdot) \) is an equilibrium profile of voting strategies for any non-empty set of candidates \( C \); and (2) \( e^* \) is an equilibrium profile of candidacy strategies, given \( \alpha^*(\cdot) \).\(^{14}\) We shall call serious contender a candidate who is in the winning set, and serious equilibrium an equilibrium where all candidates are serious contenders. Likewise, we shall call spoiler a candidate who is not in the winning set, and spoiler equilibrium an equilibrium where at least one candidate is a spoiler.

Following previous contributions in the citizen-candidate literature—like, for example, Besley and Coate (1997) or Dhillon and Lockwood (2002)—we shall focus on equilibria in pure strategies.

4. POLICY MODERATION

In this section, we compare the extent of policy moderation under the different MVs and OVs in the context of serious races, where moderation is defined relative to the median voter’s ideal policy.\(^{15}\) But before we proceed, two things must be done. First, we need to specify how we are going to compare equilibria. Since our

\(^{12}\)For example, if the election is held under PV, a citizen’s voting strategy is weakly undominated if he does not vote for the candidate(s) he likes the least. Instead, if the election is held under Approval Voting, his voting strategy is weakly undominated if he votes for the candidate(s) he likes the most and does not vote for the candidate(s) he likes the least. And if the election is held under the Borda Count, he must rank the candidate(s) he likes the most above the candidate(s) he likes the least.

\(^{13}\)Note that this assumption is not key to the results, and is made only to simplify the analysis.

\(^{14}\)Note that a political equilibrium exists since action spaces are finite.

\(^{15}\)We focus on serious races because of equilibrium characterization concerns. See Section 5.2 of the paper for a discussion of the other class of elections, namely, spoiler races.
focus is on policy outcomes, we shall say that two equilibria are strongly equivalent if the lotteries over the policy outcomes are the same—i.e., the probability a policy is implemented is the same in each of the two equilibria. Alternatively, we shall say that two equilibria are equivalent if the policies that are implemented with a strictly positive probability are the same in both equilibria.\(^{16}\) Second, if we want to study the effect of allowing citizens to vote for several candidates instead of only one, we first need to characterize the equilibrium set under PV. From Besley and Coate (1997) we know there are only two types of serious equilibria, namely, the one- and two-position equilibria. In the one-position equilibria, only one citizen chooses to stand for election. In the two-position equilibria, two citizens choose to run, one on each side of the median citizen’s ideal policy, with the median citizen indifferent between them. Note that the one-position equilibrium set is strongly equivalent under any MV and OV.

### 4.1. Allowing for Vote Cumulation

We first consider elections where cumulated ballots are admissible. In this context, a citizen can cast all his votes for one candidate or he can split them between several candidates as he wishes.

We can then establish:

**Proposition 1.** Suppose cumulated ballots are admissible. Then, the serious equilibrium sets under any two MVs and OV are strongly equivalent. \(||

That a citizen is pivotal for every serious contender is key to understand this result. It implies that a citizen should stack all his votes on one serious contender instead of wasting any of them on another candidate.\(^{17}\) Consequently, giving citizens more than one vote to cast would only scale up by a common factor the score of every candidate without eliminating the wasting-the-vote phenomenon. The equilibrium set would therefore remain unchanged compared to PV.\(^{18}\)

### 4.2. Multiple Vote Scoring Rules

In this section, we compare equilibria under the different MVs. A MV is a voting rule where a citizen is given \(q \in \mathbb{N}\) votes to cast—or \((c - 1)\) votes if the number of candidates \(c\) does not exceed \(q\)—and each vote is worth one point. Formally, the vector of points a voter can cast \(s = (s_1, \ldots, s_c)\) is given by

\[
\begin{align*}
1 = s_1 = \ldots = s_q > s_{q+1} = \ldots = s_c = 0 & \quad \text{if } c > q; \\
1 = s_1 = \ldots = s_{c-1} > s_c = 0 & \quad \text{otherwise}.
\end{align*}
\]

Examples of MVs are PV where \(q = 1\), Dual Voting where \(q = 2\), and Negative Voting (or Approval Voting if truncated ballots are admissible) where each citizen is given a number of votes equal to the number of candidates less one (or equivalently, \(q = +\infty\)).

Let us first consider elections where only completely-filled ballots are admissible.

\(^{16}\)Observe that equivalence does not require that the probability a policy is implemented is the same in each of the two equilibria. Thus, strong equivalence implies equivalence, but not conversely.

\(^{17}\)See Lemma 4 in Dellis (2006) for a formal statement and proof of this result.

\(^{18}\)It is worth noting that with a different model, Gutowski and Georges (1993) get the same equivalence result.
Proposition 2. Suppose only completely-filled ballots are admissible. Then, for any serious equilibrium under PV there exists a strongly equivalent serious equilibrium under any MV. \[\Box\]

To understand this result, recall that when the election is held under PV, in any serious equilibrium there are candidates on at most two platforms, with only one candidate standing on each. That there is strong equivalence in the case of one-position equilibria was noted above. To see why the same holds true for the two-position equilibria, note that although a MV with \(q \geq 2\) cannot deter multiple candidacies at the winning positions, it can accommodate them—in the sense that all the citizens who end up running for election can tie for first place.\(^{19}\) At the same time, it can deter entry by both moderates (i.e., citizens whose ideal policies lie between the two winning platforms) and extremists (i.e., citizens for whom one of the winning policies lies between their ideal one and the median). This is illustrated most simply in the case of Dual Voting.

Example 1. Consider a community which has to decide on a tax rate. There are 39 citizens whose ideal tax rates are distributed on the set \(\{0, 1, \ldots, 10\}\). Let there be seven citizens with ideal rate 5, \((6 - k)\) citizens with ideal rate \(5 \pm k\) (for \(k = 1, \ldots, 4\)) and two citizens with ideal rates 0 and 10, respectively. Citizens' preferences are represented by a quadratic loss utility function; that is, \(u^k(x) = -(x - x)^2\). Finally, let \(\delta = 3\) be the utility cost of running for election.

Suppose first that the election is held under PV. Denote by \(\{x, y\}\) an equilibrium where candidates are standing on platforms \(x\) and \(y\). Then \(\{3, 7\}, \{2, 8\}, \{1, 9\}\) and \(\{0, 10\}\) are all two-position equilibria, with one candidate standing on each platform. All the citizens at 0, \(\ldots, 4\) (6, \(\ldots, 10\), resp.) vote for the left (right, resp.) candidate and the citizens at 5 abstain. Hence, both candidates tie for first place. Moreover, no other citizen is willing to enter the race if he (correctly) anticipates that voters will fear wasting their vote on him.

Suppose now that the election is held under Dual Voting. The same lotteries arise, albeit with two candidates standing on each platform. To see this, consider the \(\{3, 7\}\) equilibrium. As under PV, at least one citizen at 3 and another one at 7 are willing to run for election. Now, let a second candidate stand at 3. All the citizens at 0, \(\ldots, 4\) cast their two votes for the candidates at 3. All the citizens at 6, \(\ldots, 10\) cast one vote for the candidate at 7 and the other one for one of the candidates at 3. All the citizens at 5 abstain. Hence, the candidate at 7 no longer ties for first place, and 3 is adopted with probability 1 (instead of 1/2). The second candidate at 3 is thus willing to run for election since the utility gain—equal to 8—is larger than the cost of running. Similarly, a second citizen will enter at 7. But once there are two candidates standing on each platform the wasting-the-vote phenomenon reappears. Those equilibria can then be sustained by the anticipation on the part of any other citizen that voters will fear wasting any of their two votes on him. \(\Box\)

Thus, Proposition 2 shows that letting citizens vote for several candidates without allowing them to truncate their vote or rank-order the candidates would expand the equilibrium set.\(^{20}\) Hence, if one is concerned about focal manipulation, this suggests it may not be the right way to go.

\(^{19}\)The proof of Proposition 2 shows that there will necessarily be at least two, but no more than \(q\), contenders on each platform (with an equal number on each).

\(^{20}\)To be precise, Proposition 2 only shows that the equilibrium set would be at least as large. However, it is easy to construct examples of equilibria under a MV with \(q \geq 2\) where serious
The above result leaves two open questions. First, it does not address the question of policy moderation. While we have not been able to formally prove that the serious equilibrium set under PV is moderate compared to the one under any other MV, it is however possible to construct equilibria where the policy outcomes lie further away from the median than in any equilibrium under PV (such an example is available from the author). Second, it does not address the question of majority representation. The following example does so, showing the possibility of lopsided elections when citizens are given three or more votes to cast.

**Example 2.** Consider the community described in Example 1, but with six citizens at 5 and three at each other position. Moreover, let the utility cost of running be \( \delta = 2 \). In that case, \{4, 6\}, \{3, 7\}, ... and \{0, 10\} with one candidate on each platform are the only two-position serious equilibria under PV. Suppose instead that the election is held under the MV where \( q = 3 \). Then, \{6, 9\} can be supported by a two-position serious equilibrium, with one candidate standing at 6 and three at 9.\(^{21}\) All the citizens at 8, 9 and 10 vote for the candidates at 9. All the citizens at 0, ..., 7 vote for the candidate at 6 and cast their other two votes for the candidates at 9 in such a way that each gets a total score of 27 points. Neither of the candidates wants to exit and no other citizen wants to enter the race if they (correctly) anticipate that this would trigger a reaction on the part of the voters that would lead to the election of a candidate running on their least-preferred platform among 6 and 9. \( \square \)

Let us now consider elections where truncated ballots are admissible. First, note that we can restrict our attention to the class of two-position serious equilibria. This is because when truncated ballots are admissible, the only possible equilibria are the one- and two-position equilibria (for a proof, see Proposition 3 in Dellis 2006) and that, as mentioned above, the one-position serious equilibrium set is strongly equivalent under any MV or OV. It is worth noting as well that the policy outcome in a one-position serious equilibrium is moderate—in the sense of being preferred by the median citizen—compared to the ones in a two-position serious equilibrium (for a proof, see Lemma 3 in the appendix). In any two-position serious equilibrium, at most \( q \) candidates are standing on a platform, and the platforms lie on either side of the median with the median citizen indifferent between them.\(^{22}\)

We then have:

\(^{21}\) Thus, in equilibrium, the extreme policy (i.e., 9) is three times more likely to be adopted than the moderate one (i.e., 6). It is worth noting that this feature is not idiosyncratic to our example. In fact, the extreme policy—say \( x_j \)—must be adopted with a probability which is \( \left( \frac{\gamma_i}{\gamma_j} \right) \) times the probability with which the moderate policy—say \( x_i \)—is implemented, where \( \gamma_i \equiv \# \{ \ell \in N : v_{i\ell}^j > v_{j\ell}^i \} \) and \( \gamma_j \) is similarly defined.

\(^{22}\) Note that, contrary to the case where only completely-filled ballots are admissible, there need not be more than one candidate standing on a platform. This is because citizens are no longer forced to vote for a candidate they dislike. This is best seen in the context of Example 1 when the election is held under Dual Voting. If a second candidate stands at 3, all the citizens at 6, ..., 10 will now vote only for the candidate at 7. This means that all three candidates are now tying—each receiving 16 votes. This implies that the second candidate at 3 is better off not running since the utility gain is equal to \( 8/3 \)—his entry in the race increasing the probability that 3 is adopted from 1/2 to 2/3 (instead of 1 when truncated ballots are admissible)—, less than the cost of running.

It is also worth mentioning that when cumulated ballots are admissible, serious contenders are located on at most two platforms as well. However, with cumulated ballots, only one serious contender can stand on each platform. Intuitively, when cumulated ballots are admissible, the most effective way for a leftist to improve the chances that the left policy is undertaken is by...
Proposition 3. Let the election be held under $V$ an arbitrary MV, with $q \in \mathbb{N}$ the maximum number of votes a citizen is ever allowed to cast under $V$. Suppose truncated ballots are admissible. Then, the serious equilibrium set under $V$ is a subset of the serious equilibrium set under PV—in the sense that for any serious equilibrium under $V$, there exists an equivalent one under PV. Moreover, there exists $\overline{q} \in \mathbb{Z}_+$ finite such that if $q > \overline{q}$ and preferences are symmetric—i.e., $u^\ell(x) \equiv u(|x_L - x|)$—, then this subset is moderate in the sense that if the lottery that implements $x_L$ and $x_R$ is a serious equilibrium outcome under $V$, then for any serious equilibrium under PV that implements $\overline{x}_L$ and $\overline{x}_R$ with $[\overline{x}_L, \overline{x}_R] \subset [x_L, x_R]$, there exists a strongly equivalent serious equilibrium under $V$.

This result shows that for any serious equilibrium under an arbitrary MV, there exists an equivalent one under PV, but not conversely. The wasting-the-vote phenomenon is what drives the result. Indeed, under PV no other citizen is willing to enter the race if he anticipates that citizens will fear wasting their vote on him. But this does not happen under the other MVs if (strictly) less than $q$ candidates are standing on each of the two platforms. A moderate candidate may then want to run for office if he anticipates he will get enough votes to win the election.

Moreover, Proposition 3 shows that if preferences are symmetric and citizens are given enough votes to cast under $V$ (i.e., $q > \overline{q}$), then the policy outcomes that are supported by equilibria under both $V$ and PV are the closest ones from the median citizen’s ideal policy. This follows from the fact that the threat of entry comes only from the moderates. The farther away from the median voter’s ideal policy the two platforms are, the more likely it is that a moderate will be able to garner enough votes to win the election and the more he will care about getting his ideal policy adopted.

Hence, Proposition 3 suggests that letting citizens vote for several candidates, allowing them to truncate their ballot but not rank-order the candidates, may improve the electoral prospects of the moderates. In turn, this reduces the size of the equilibrium set—and thus the possibility of focal manipulation—and may yield policy moderation. However appealing this may sound, three qualifications...
must be kept in mind. First, the equilibrium set need not be significantly different from the one obtained under PV.\textsuperscript{27} Second, if preferences are not symmetric, then extremism instead of moderation may be the result. Indeed, when citizens differ on how intense their policy preferences are, we can end up with more citizens standing on one platform than on the other. It is then more likely that the former policy will be undertaken. And if the two positions are not too far apart, the utility gain from running on the latter platform may then be lower than its cost.\textsuperscript{28} Finally, if citizens are given too few votes to cast (i.e., $q \leq \bar{q}$), extremism may arise as well. The next example sheds some light on the latter qualification.

Example 3. Consider the community described in Example 1, except that truncated ballots are now admissible. Then, \{3, 7\} can no longer be supported as an equilibrium outcome under Dual Voting. To see why, observe that running for election is costly enough that only one candidate wants to stand on each platform (see footnote 22). Consequently, a citizen at 5 entering the race will receive a vote from all the citizens whose ideal tax rate is 4, 5 or 6 (by weak undominance). Hence, his vote total is at least 17. At the same time, the candidate at 3 (7, resp.), being the least-preferred candidate of all the citizens with an ideal tax rate of 5 or higher (lower, resp.), does not get any vote from all those citizens. Hence, their vote totals are at most 16. Therefore, the candidate at 5 wins outright. And since his utility gain from getting his most-preferred tax rate—which equals 4—is larger than the cost of running—which equals 3—, he thus enters the race.

At the same time, \{2, 8\}, \{1, 9\} and \{0, 10\} are still all two-position equilibria. To see why, note that for each, the two policies are far enough apart that two citizens want to stand on each platform. Hence, the wasting-the-vote phenomenon is back, and no other citizen is thus willing to enter the race if he (correctly) anticipates that citizens will fear wasting one of their two votes on him.\textsuperscript{29}

The above example shows that if citizens are not given enough votes to cast, then the wasting-the-vote phenomenon may reappear, thus preventing entry by moderates. And this is more likely to occur when extremists are running since the policies being then far apart, more candidates should be willing to run on each of the two platforms.

4.3. Ordinal Vote Scoring Rules

In this section, we compare equilibria under the different OVs. An OV is a voting rule where a citizen is asked to rank-order the candidates. The candidate who is ranked $h^{th}$ on the ballot gets $s_h$ point, where the vector of points is given by $1 = s_1 > s_2 \geq ... \geq s_c = 0$. Examples of OVs are PV where $s_1 = 1$ and $s_2 = ... = s_c = 0$, or the Borda Count where $s_h = \frac{c-h}{c-1}$ (for $h = 1, ..., c$).

As for the MVs with truncated ballots admissible, we can again restrict our attention to the two-position equilibria.\textsuperscript{29} In those equilibria, there is one candidate position equilibria arising under PV—i.e., \{3, 7\}, \{2, 8\}, \{1, 9\} and \{0, 10\}—are serious equilibria if the election is held under Approval Voting. Moreover, it is worth noting that from a normative point of view, Approval Voting dominates PV.

\textsuperscript{27}To see this, consider Example 1 but with $\delta = 5/2$. Then, the serious equilibrium set under Dual Voting is strongly equivalent to the equilibrium set under PV.

\textsuperscript{28}This qualification has already been identified by Dellis and Oak (2006) for the case of Approval Voting (i.e., $q = +\infty$)—see Example 3 in their paper. We find here that it generalizes to other MVs with $q \geq 2$. An example for the case of Dual Voting is available from the author.

\textsuperscript{29}Indeed, in equilibria where voters have only one top-score vote to cast (i.e., $s_1 \neq s_2$), there
Proposition 4. Let $V$ and $\tilde{V}$ be any two OVs, and denote by $s$ and $\tilde{s}$ their respective second-place scores in three-way races. Suppose $s > \tilde{s}$. Then, the serious equilibrium set under $V$ is a moderate subset of the serious equilibrium set under $\tilde{V}$, where moderation is defined as in Proposition 3.

Thus, the equilibrium set shrinks as the second-place score in a three-way race increases. This is because the bigger the second-place score is, the less effective an OV is in deterring entry. To be more precise, suppose first that only completely-filled ballots are admissible, and let us start by considering the issue of entry at the winning positions. A second citizen can be deterred from entering on any of those platforms only if the loss from being ranked on a voter’s ballot second instead of first is larger than the gain from being ranked second instead of third and last. To see this, suppose a second candidate stands on the left platform. Then the leftists will rank the right candidate last while the rightists will have no other choice than ranking second one of the two left candidates. Thus, only a left candidate can be elected if $s$ is large. Anticipating this, a second leftist will then want to enter the race. Thus, we can distinguish two types of OVs: those with $s > \frac{1}{2} \left( \frac{N-M-2}{N-M-1} \right)$ (that we shall call worst-punishing) and the others (that we shall call best-rewarding). The latter OVs can effectively deter multiple entries at the winning positions, not the former ones. Consequently, the serious equilibrium set under a worst-punishing OV contains only one-position equilibria. Thus, it only remains to consider the issue of entry by extremists and moderates in the context of elections held under best-rewarding OVs. The proof of Proposition 4 shows that those voting rules can be fully effective in discouraging extremists from entering the race. Moreover, a moderate does not want to stand for office if the score loss of being ranked second instead of first, $(1 - s)$, is large—he anticipating that the leftists (rightists, resp.) will rank the left (right, resp.) candidate first and him second because they fear wasting their top-score vote on him.

are at most two winning positions (see Proposition 1 in Dellis 2006). Moreover, the one-position serious equilibria are strongly equivalent under any MV and OV (as noted above) and are moderate compared to the two-position serious equilibria (as shown in Lemma 3).

Recalling that PV is the OV where $s = 0$, we then get as a corollary to Proposition 4 that the serious equilibrium set under any OV is a moderate subset of the serious equilibrium set under PV.

Note that the same intuition explains why there are at least two candidates standing on each platform in an election held under a MV (with $q \geq 2$) and where only completely-filled ballots are admissible. While it does not preclude the existence of a two-position serious equilibrium under a MV, it does under an OV with a sufficiently large $s$. This is because MVs can accommodate more than one serious contender on a platform, but OVs cannot.

This classification is similar to the one introduced by Cox (1987), except for the term in brackets. This difference is due to strategic voting.

It is worth noting that for those OVs—among which lies the Borda Count—there is a unique serious equilibrium when the cost of running for election $\delta$ is sufficiently small. In that equilibrium, the median citizen—and thus the Condorcet winner—runs unopposed. This provides a theoretical underpinning for the argument often made that the Borda Count elects the Condorcet winner more often than the other commonly-discussed voting rules.
However, entry at the winning positions is no longer an issue when truncated ballots are admissible. This is because citizens can (and will) then abstain from putting on their ballot a candidate they dislike. To see this, consider again the situation where a second candidate stands on the left platform. Since vote truncation is allowed, the rightists will then choose to rank only the right candidate. Consequently, either one of the two left candidates will tie with the right candidate, or the latter will win outright. No second leftist is thus willing to stand for election. This implies that we must consider only the possibility of entry by moderates. Not surprisingly, it is more likely to occur the bigger \( s \) is and the more polarized the platforms are.

Besides, Proposition 4 shows that the most extreme equilibria are the ones that do not survive as \( s \) increases. The reason is that either the two-position equilibrium set is empty—as under the worst-punishing OVs with only completely-filled ballots admissible—, or the threat of entry comes only from the moderates.

To sum up, Proposition 4 suggests that letting citizens rank-order the candidates would both reduce the size of the equilibrium set—and thus the possibility of focal manipulation—and yields policy moderation.\(^{35}\) However, the extent of the set size reduction as well as the degree of policy moderation need not be substantial. To appreciate this, consider the following example.

**Example 4.** Consider the community described in Example 1. Recall that if the election is held under PV, then \( \{3, 7\}, \{2, 8\}, \{1, 9\} \) and \( \{0, 10\} \) are all two-position equilibria, with one candidate standing on each platform. Suppose instead that the election is held under an OV with \( s = 1/3 \) and that citizens are allowed to truncate their ballots. Then, the serious equilibrium set is strongly equivalent to the one under PV.\(^{36}\) To see this, take the \( \{0, 10\} \) equilibrium. Those policies being far apart, neither of the two candidates would be better off not running for election. At the same time, the other citizen at 0 does not want to enter the race since at best he will be ranked second by all the leftists (i.e., citizens at 0, ..., 4) — thus leaving the policy outcome unchanged — and at worst he will be ranked first by some — thus resulting in the outright victory of the candidate at 10. The same is true for the other citizen at 10. Finally, no other citizen wants to stand for election if they (correctly) anticipate that they will be ranked only by the citizens for whom they are the most-preferred candidate and that the leftists (rightists, resp.) will still top-rank the candidate at 0 (10, resp.). For example, a candidate at 5 would then be ranked first by all the citizens at 5 and second by all the citizens at 3, 4, 6 and 7. He would thus get 13 points, while the candidates at 0 and 10 would get each 16 points.\(^{37}\) \( \square \)

The next proposition looks at the effect of allowing for vote truncation.\(^{38}\)

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\(^{35}\)It is worth noting that OVs with truncated ballots admissible yield policy moderation whether preferences are symmetric or not. This contrasts with MVs. The intuition behind this difference lies in the (in)ability to accommodate more than one serious contender on a platform. OVs are unable to do so since a second candidate may split the top-score votes among those two candidates. MVs (with \( q \geq 2 \)) are able to do so since there is no such risk of vote splitting. But, as noted earlier, if preferences are asymmetric, then under a MV, entry at one position can be so large that nobody finds it profitable to stand on the other one.

\(^{36}\)Note that the same is true if only completely-filled ballots were admissible.

\(^{37}\)Note however that the extent of policy moderation would be more substantial for a bigger \( s \). For example, only \( \{3, 7\} \) and \( \{2, 8\} \) can be supported as two-position serious equilibria if the election is held under the Borda Count with truncated ballots admissible, and none if only completely-filled ballots are admissible.
Proposition 5. Let the election be held under $V$ an arbitrary OV. If $V$ is best-rewarding, then the serious equilibrium set when truncated ballots are admissible is a moderate subset of the serious equilibrium set when only completely-filled ballots are admissible, where moderation is defined as in Proposition 3. Instead, if $V$ is worst-punishing, then the serious equilibrium set when only completely-filled ballots are admissible is a moderate subset of the serious equilibrium set when truncated ballots are admissible.

Thus, allowing for vote truncation reduces the possibility of focal manipulation and yields policy moderation if the election is held under a best-rewarding OV. The reverse holds true if instead, the election is held under a worst-punishing OV. To understand the latter, recall that worst-punishing OVs are unable to deter and accommodate multiple similar candidacies when only completely-filled ballots are admissible. The two-position serious equilibrium set is then empty. Hence the result. To understand the former, note that when the election is held under a best-rewarding OV, allowing for vote truncation makes it easier for a moderate candidate to get a higher vote share. Loosely interpreted, this is because extremists fear their most-preferred candidate may be defeated by the moderate candidate. When vote truncation is allowed, their best strategy is then to rank only their most-preferred candidate. Instead, when vote truncation is not allowed, their best strategy is to rank the moderate candidate last. Hence, the moderate candidate gets the same vote total in both cases while each of the other candidates gets a higher vote total when vote truncation is not allowed than when it is.

4.4. Comparing MVs and OVs

We now compare the size of the equilibrium set and the extent of policy moderation under the different MVs and OVs.

Let us first consider elections where only completely-filled ballots are admissible. It follows from our previous results that on the one side, we have worst-punishing OVs which equilibrium sets contain only one-position equilibria and are, therefore, the smallest and most moderate ones. On the other side, we have the MVs, which equilibrium sets are supersets of the equilibrium set obtained under PV. Finally, best-rewarding OVs (as well as MVs and OVs with vote truncation allowed) lie in-between.

Let us now consider elections where truncated ballots are admissible. The next proposition shows that when policy preferences are symmetric, a MV that gives each citizen enough votes to cast yields a smaller and more moderate serious equilibrium set than any OV. Formally,

Proposition 6. Let $V$ and $\tilde{V}$ be an arbitrary MV and OV, respectively. Suppose preferences are symmetric and truncated ballots are admissible. Let $q \in \mathbb{N}$ denote the maximum number of votes a citizen is ever allowed to cast under $V$ and define $\overline{q}$ as in Proposition 3. Then, if $q > \overline{q}$, the serious equilibrium set under $V$ is a moderate subset of the serious equilibrium set under $\tilde{V}$, where moderation is defined as in Proposition 3.

To understand the intuition underlying this result, recall that when vote truncation is allowed differences in serious equilibrium sets depend only on whether entry between the two platforms can be deterred in two-position races. A moderate is deterred from entering the race if either it is relatively too costly to do so, or the
electoral outcome would remain unchanged. If the former holds true, it does so differently under any MV and OV.\textsuperscript{38} The same need not be true for the latter. Indeed, under a MV with $q > \frac{7}{1}$, a moderate receives a vote—and thus one full point—from all the citizens for whom he is the most-preferred candidate. Under an OV, he may however receive less than one point from some of those voters—they ranking him second on their ballot if they fear wasting their top-score vote. Hence, the minimal score a moderate can get if he enters the race is larger under a MV with $q > \frac{7}{1}$ than under any OV. The former voting rules are thus more effective at lowering the barriers to entry by moderates than the latter ones.

However, Proposition 6 is silent about the cases of non-symmetric preferences and of MVs with $q \leq \frac{7}{1}$. As we know from the discussion that follows Proposition 3, the equilibrium set may then be an extreme subset of the equilibrium set under PV. By transitivity, this extends to the comparison with OVs—the latter equilibrium sets being moderate subsets of the equilibrium set under PV.

5. DISCUSSION

We now briefly discuss two extensions of the previous analysis.

5.1. Refining the Voting Behavior

When truncated ballots are admissible, the weak undominance refinement may be too weak.\textsuperscript{39} We now propose a stronger refinement, that we call Relative Undominance.

**Definition 3 (Relative Undominance).** Let $\mathcal{C}$ be a non-empty set of candidates and $\alpha (\mathcal{C})$ a profile of weakly undominated voting strategies. Suppose truncated ballots are admissible. Citizen $\ell$’s voting strategy $\alpha^\ell (\mathcal{C})$ is said to be undominated relative to others’ voting strategies if:
(1) $\alpha^\ell_i (\mathcal{C}) = 0$ for all candidate $i$ such that $v^\ell_i < V^\ell (\mathcal{C}, \alpha)$; and
(2) there does not exist $\alpha^\ell_\ell (\mathcal{C})$ such that $\alpha^\ell_i (\mathcal{C}) \geq \alpha^\ell_\ell (\mathcal{C})$ for all candidate $i$ with $v^\ell_i > V^\ell (\mathcal{C}, \alpha)$, with a strict inequality for at least one such candidate $i$.

Condition (1) requires that a citizen does not vote for a candidate he likes strictly less than the winning lottery. Condition (2) requires that a citizen votes for as many of the candidates he strictly prefers to the winning lottery.\textsuperscript{40}

The following lemma proves the existence of voting equilibria in relatively undominated pure strategies.

\textsuperscript{38}Note that this is true because, preferences being symmetric, each policy is adopted with an equal probability.

\textsuperscript{39}To see this, suppose three candidates are standing for election—say, a left, a moderate and a right candidate. All the leftists (rightists, resp.) prefer the left (right, resp.) candidate and are almost indifferent between the left (right, resp.) and the moderate candidates but they very much dislike the right (left, resp.) candidate. It would thus seem natural that the moderate candidate should be on everybody’s ballot if voting for several candidates was allowed. But this need not be the case under weak undominance. For example, all the leftists voting only for the left candidate and all the rightists voting only for the right candidate can be a weakly undominated voting equilibrium.

\textsuperscript{40}Two comments are worth being made. First, Relative Undominance is equivalent to the Relative Sincerity refinement proposed by Dellis and Oak (2006) for elections held under Approval Voting. Second, Laslier (2005) shows that relatively undominated voting strategies are optimal when the election is held under Approval Voting, the electorate is large and the distribution of voters’ policy preferences uncertain.
Lemma 2. Let the election be held under an arbitrary MV or OV, and suppose that truncated ballots are admissible. For any non-empty set of candidates, a voting equilibrium in relatively undominated pure strategies exists. ||

We can then establish:

Proposition 7. Consider an election where truncated ballots are admissible. Suppose first that the election is held under an arbitrary MV, with \( q \in \mathbb{N} \) the maximum number of votes a citizen is ever allowed to cast under this voting rule. There exists \( q^* \in \mathbb{Z}_+ \) finite such that if \( q > q^* \), then in any relatively undominated serious equilibrium, we have \( -v^m_i < \delta \) for every candidate \( i \).\(^{41}\)

Suppose instead that the election is held under an arbitrary OV. Let \( V \) and \( \tilde{V} \) be any two OVs, with \( s \) and \( \tilde{s} \) denoting their respective second-place scores in three-way races. If \( s > \tilde{s} \), then the relatively undominated serious equilibrium set under \( V \) is a subset of the relatively undominated serious equilibrium set under \( \tilde{V} \). Moreover, there exist \( \underline{s} \) and \( \overline{s} \) with \( 0 < \underline{s} < \overline{s} < 1 \) such that for \( V \) an arbitrary OV, we have that: (1) if \( s < \underline{s} \), then the relatively undominated serious equilibrium set is strongly equivalent to the serious equilibrium set; and (2) if \( s > \overline{s} \), then in any relatively undominated serious equilibrium, we have \( -v^m_i < \delta \) for every candidate \( i \).\(^{42}\) ||

Thus, Proposition 7 extends Propositions 3 and 4 to the case where citizens vote in a relatively undominated way.\(^{43}\) In addition, it shows that the degree of policy moderation can be substantial, the policy outcomes being close enough to the ideal policy of the median citizen that he does not want to enter the race (i.e., \( -v^m_i < \delta \)). However, this holds true under the conditions that the MV gives each citizen enough votes to cast (otherwise the wasting-the-vote phenomenon may reappear and yield extremism instead of moderation) and that the OV has a second-place score which is large enough (the Relative Undominance refinement has no bite if \( s < \underline{s} \)).\(^{44}\)

5.2. Spoiler Equilibria

So far, we have only considered serious equilibria. We now examine spoiler equilibria; that is, equilibria where some candidates are running not to win the election but because their presence in the race yields a different policy outcome.\(^{45}\)

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\(^{41}\)Note that \( q^* = \overline{s} \) if preferences are symmetric, where \( \overline{s} \) is defined as in Proposition 3.

\(^{42}\)Note that \(-u^m(x_0) \geq \delta\)—i.e., the median citizen willing to run against the status-quo—is sufficient for the existence of a relatively undominated serious equilibrium where \(-v^m_i < \delta\) for every candidate \( i \). Note also that \( \underline{s} \) and \( \overline{s} \) are not tight. Indeed, it is possible to construct examples where \( s > \underline{s} \) and the Relative Undominance refinement has no bite, or \( s < \overline{s} \) and \(-v^m_i < \delta\) for every candidate \( i \). Examples are available from the author.

\(^{43}\)It is worth noting that in Proposition 3, the moderation result was obtained under the condition that policy preferences are symmetric. However, we can dispense with that condition here. Intuitively, a MV that gives citizens enough votes to cast is ineffective in deterring entries at the winning positions. But whether it does matter depends on the form of preferences. If preferences are symmetric, then the same number of citizens are willing to stand on each platform and, therefore, only the entry by moderates matters. Instead, if preferences are not symmetric a larger number of citizens may be willing to stand on one platform, up to a point where the utility gain from running on the other platform becomes so scant that nobody wants to do so. The entry at the winning positions then matters. This is no longer the case however if citizens vote in a relatively undominated way since then the entry by moderates becomes the dominant force.

\(^{44}\)That the Relative Undominance refinement has no bite when \( s < \underline{s} \) is very intuitive. A citizen does not want to enter the race if he anticipates that voters will continue to top-rank the same candidate, while ranking him second; a low second-place score does not leave him any prospect of victory.

\(^{45}\)Whether spoiler candidates exist in practice can be debated.
Because of characterization concerns, we shall limit ourselves to revisit two of our previous results.

The first one concerns the issue of representativeness. In elections held under a MV with \( q \geq 3 \) and only completely-filled ballots admissible, serious equilibria exist where all the candidates are standing on the same side of the median. In the context of serious races, only those voting rules exhibit this feature. However, in the context of spoiler races, the same can happen if the election is held under an OV and only completely-filled ballots are admissible. This has important implications. Indeed, we have argued that the worst-punishing OVs without vote truncation are the ones that perform the best in terms of policy moderation and equilibrium multiplicity (even if we restrict ourselves to relatively undominated equilibria). This conclusion is brought into question if some citizens are willing to stand as spoilers.

The second one concerns the extent of policy moderation in relatively undominated equilibria. We know from Proposition 7 that it is substantial if citizens are given enough votes to cast when the election is held under a MV, or the second-place score is large enough when the election is held under an OV. However, this result does not extend to spoiler races. Intuitively, candidates are deterred from stepping down and potential candidates from entering the race if they anticipate a reaction on the part of voters that will ultimately lead to a less-preferred electoral outcome. The latter more easily happens when there are spoilers since there are more positions at which candidates are standing.

The following example sheds some light on this discussion.

**Example 5.** Consider the community described in Example 1, but with 44 citizens whose ideal policies are uniformly distributed on the set \( \{0, 1, \ldots, 10\} \). Let the status-quo policy \( x_0 = 0 \) (i.e., the community considers introducing a new tax). Suppose the election is held under Approval Voting. Thus, a citizen at 4, 5 or 6 running unopposed are the only serious equilibria if citizens vote in a relatively undominated way.

Consider now the following situation. Let there be four candidates, standing at 0, 1, 2 and 4, respectively. Let all the citizens at 0 and 1 vote for the candidates at 0, 1 and 2; the citizens at 2 vote only for the candidate at 2; and the citizens at 3, ..., 10 vote for the candidates at 2 and 4. Hence, candidate 2 wins outright. Now, the candidate at 0 does not want to exit the race if he (correctly) anticipates that all the citizens at 0, 1 and 2 will then vote for the candidates at 1 and 2; the citizens at 3 for the candidates at 2 and 4; and the citizens at 4, ..., 10 only for the candidate at 4. Similarly for the candidate at 1 (2, resp.) if he anticipates that all the citizens at 0 and 1 will vote for the candidates at 0 and 2 (0 and 1, resp.); the citizens at 2 only for the candidate at 2 (1, resp.); the citizens at 3 for the candidates at 2 and 4 (4, resp.); and the citizens at 4, ..., 10 only for the candidate at 4. Finally, the candidate at 4 does not want to exit the race if he anticipates that all the citizens will vote for the candidate at 1, while the citizens at 0 (2, ..., 10, resp.) will also cast a vote for the candidate at 0 (2, resp.). And no other citizen is willing to enter the race if he anticipates that the candidate at 2 will still receive the votes of all citizens.

Hence, the above is a relatively undominated spoiler equilibrium where \( -v_i^m > \delta \).

Note that the same applies to OVs. To see this, consider the same community as above but with the election held under the Borda Count and with ten citizens at 5. Hence, \( s = \frac{5}{9} > \frac{1}{10} = \pi \) and by Proposition 7 we know that \( -v_i^m < \delta \) for all candidate \( i \) in any relatively undominated serious equilibrium. However,
\[ \mathcal{C} = \{0, 1, 2, 4\} \text{ with } W(\mathcal{C}, \alpha) = \{2\} \] can be supported by a relatively undominated spoiler equilibrium under the Borda Count as well. \( \square \)

6. CONCLUSION

In this paper, we have looked at the claim that letting people vote for several candidates would improve the electoral prospects of the moderates and, therefore, yield policy moderation. In contrast to the previous literature, we have done so in a setting that allows for endogenous candidate entry, strategic voting and policy-motivated candidates—all features that have received both empirical and experimental support. This is important given that some of our conclusions are in sharp contrast with what had been previously thought. Moreover, we have investigated different ways in which to let people express their preferences for the different candidates and shown that their implications for policy moderation can be entirely opposite.

Our results show that letting people vote for several candidates need not yield policy moderation if: (1) citizens are allowed to cumulate their votes—which sole effect is to scale up by a common factor the vote total of each candidate; (2) each citizen is given several votes, all of which must be cast for the ballot to be valid—which results in an expansion of the equilibrium set, the possible destruction of the two-party system with serious contenders located at more than two positions, and the possibility of lopsided elections with only a minority of the electorate represented; (3) citizens are allowed to truncate their vote but either are not given enough votes to cast—in which case the wasting-the-vote phenomenon may reappear—or have non-symmetric policy preferences—which may result in excessive entry at one position, thus discouraging anybody from standing on the other one—, both potentially yielding extremism; or (4) spoilers are running—which creates barriers to both entry and exit.

Our results confirm the above claim when all the candidates are serious contenders and: (1) citizens are given enough votes to cast and truncated ballots are admissible and either citizens have symmetric policy preferences, or vote in a Relatively Undominated way; or (2) citizens are asked to rank-order the candidates. Furthermore, the extent of policy moderation is maximal in the latter case where the second-place score is large and only completely-filled ballots are admissible (like under the Borda Count)—with the possibility of always electing the Condorcet winner if running for election is not too costly.

The present analysis has several limitations that deserve further research. First of all, we have assumed that candidates are only policy-motivated. Allowing for a mix of policy- and office-motivation is of interest. However, this extension is not trivial. The complication lies in the discontinuity that comes from the fact that a candidate will no longer be indifferent if another candidate running on the same platform wins the election instead of him. A second shortcoming is the assumption that information is perfect and complete. Relaxing this assumption in the citizen-candidate framework has yet to be done. Moreover, it would be interesting to extend the analysis to multi-dimensional policy spaces and to other electoral systems like, for example, multi-seat or runoff elections.
REFERENCES


7. APPENDIX

Let us first introduce some extra notation. Define $N_L \equiv \{ \ell \in \mathcal{N} : x_\ell < m \}$ the set of leftists, and $N_R \equiv \{ \ell \in \mathcal{N} : x_\ell > m \}$ the set of rightists. Denote by $\mathcal{M} \equiv \{ \ell \in \mathcal{N} : x_\ell = m \}$ the set of citizens who share the same ideal policy as the median citizen. Also, for any non-empty set of candidates $\mathcal{C}$ and every citizen $\ell$, let $G^\ell(\mathcal{C}) \equiv \{ i \in \mathcal{C} : v_i^\ell \geq v_j^\ell \text{ for all } j \in \mathcal{C} \}$ be the set of citizen $\ell$’s most-preferred candidates. Similarly, let $L^\ell(\mathcal{C}) \equiv \{ i \in \mathcal{C} : v_i^\ell \leq v_j^\ell \text{ for all } j \in \mathcal{C} \}$ be the set of citizen $\ell$’s least-preferred candidates. Finally, let $n_h \equiv \# \{ \ell \in \mathcal{N} : x_\ell = x_h \}$ and $c_h \equiv \# \{ i \in \mathcal{C} : x_i = x_h \}$ be the numbers of citizens and candidates at $x_h$, respectively.

We now state and prove an additional lemma. It shows that when the election is held under an OV or vote truncation is allowed, the 1-position serious equilibrium set is moderate compared to the 2-position serious equilibrium set.

**Lemma 3.** Let the election be held under $V$, and suppose that either $V$ is an arbitrary MV and vote truncation is allowed, or $V$ is an arbitrary OV. Take $(\mathcal{C}_1, \alpha)$ a 1-position serious equilibrium, and let $x_1$ denote the policy outcome. Take $(\mathcal{C}_2, \alpha)$ a 2-position serious equilibrium, and let $\{x_L, x_R\}$ be the policy outcome. Then $u^m(x_1) > u^m(x_h)$ for $h = L, R$. ||
Proof of Lemma 3. Let \( x_1 \leq m \) and \( x_L < x_R \), and note that in any serious equilibrium we must have \( u^m(x_L) = u^m(x_R) \). Assume by way of contradiction that \( u^m(x_h) \geq u^m(x_1) \) for \( h = L, R \). Hence \( x_1 \leq x_L \). Since \((C_2, \alpha)\) is an equilibrium, it must be that a candidate standing at \( x_R \) is better off running for election. It must then be that \( -u^R(x_L) \geq \delta \). At the same time \((C_1, \alpha)\) an equilibrium implies that this citizen would be worse off entering the race. This cannot be the case if \( x_1 = x_L \) since \((C_2, \alpha)\) is an equilibrium. Hence it must be that \( x_1 < x_L \). But then \( u^m(x_R) > u^m(x_1) \), and our citizen at \( x_R \) would win outright if he enters the race. It must then be that \( -u^R(x_1) < \delta \), which contradicts \( -u^R(x_L) \geq \delta \) and \( x_1 < x_L \).

We now turn to proving the lemmas and propositions stated in the text.\(^{46}\)

Proof of Proposition 1. Let \( V \) be an arbitrary MV or OV, and suppose that vote cumulation is allowed. First, note that we only need to consider the class of 2-position serious equilibria. Indeed, Dellis (2006) shows that in any equilibrium there are at most two winning positions, and that the 1-position serious equilibrium sets are strongly equivalent under any MV and OV. The following claim characterizes the set of 2-position serious equilibria under \( V \).

**Claim 1.1.** Suppose citizens are allowed to fully cumulate their votes. There exists a two-position serious equilibrium under \( V \) if, and only if,

1. there exist two citizens, say \( L \) and \( R \), whose ideal policies are \( x_L \) and \( x_R \), with \( x_L < m < x_R \) and \( v^m_L = v^m_R \); and

2. \( -\frac{v^R}{h} \geq \delta \) and \( -\frac{v^L}{h} \geq \delta \).

**Proof of Claim 1.1.** (Necessity) Condition (1) follows from Proposition 2 in Dellis (2006). Condition (2) is necessary for the two candidates to be willing to stand for election.

(Sufficiency) Let \( e \) be a vector of entry decisions, with \( e_L = e_R = 1 \) and \( e_i = 0 \) for all \( i \in N, i \neq L, R \). Thus, the only weakly undominated equilibrium profile of voting strategies is:

\[
\alpha^\ell (C) = \begin{cases} 
(1,0) & \text{for all } \ell \in N_L \\
(0,0) & \text{for all } \ell \in M \\
(0,1) & \text{for all } \ell \in N_R 
\end{cases}
\]

, where \( \alpha^\ell (C) \equiv (\alpha^\ell_L, \alpha^\ell_R) \). Hence, \( W(C, \alpha) = C \). Take \( k \in N, k \neq L, R \). Construct \( \tilde{\ell} \) such that \( \tilde{\ell}_k = 1 \) and \( \tilde{\ell}_i = e_i \) for all \( i \in N, i \neq k \). Denote by \( \tilde{C} \) the set of candidates associated with \( \tilde{\ell} \). Construct \( \alpha(\tilde{C}) \) such that:

\[
\alpha^\ell (\tilde{C}) = \begin{cases} 
(1 + s_2, 0, 0) & \text{for all } \ell \in N_L \\
(0, 0, 1 + s_2) & \text{for all } \ell \in N_R 
\end{cases}
\]

, and for all \( \ell \in M 

\[
\alpha^\ell (\tilde{C}) = \begin{cases} 
(0, 0, 1 + s_2) & \text{if } x_k < x_L \\
(0, 0, 0) & \text{if } x_k \in \{x_L, x_R\} \\
(0, 1 + s_2, 0) & \text{if } x_L < x_k < x_R \\
(1 + s_2, 0, 0) & \text{if } x_k > x_R 
\end{cases}
\]

\(^{46}\)For the sake of brevity, some of the details are omitted (they are available from the author). Moreover, the number of citizens on either side of the median citizens’ ideal policy - i.e. \( \#N_h \), \( h = L, R \) - is assumed to be even. This assumption is w.l.o.g., and is only made to simplify notation.
, where \( s_2 \in [0, 1] \) denote the second-place score in a 3-way race and \( \alpha^L(\tilde{C}) \equiv (\alpha^L_L, \alpha^L_R) \). Hence, \( W(\tilde{C}, \alpha) = \mathcal{C} \) if \( x_k \in [x_L, x_R] \}, \{R\} \) if \( x_k < x_L \) and \{L\} if \( x_k > x_R \). Thus, \( U^k(\tilde{C}, \alpha) < U^k(\mathcal{C}, \alpha) \).

Finally, let \( \alpha(\tilde{C}) \) be any voting equilibrium for every other set of candidates \( \tilde{C} \).

It follows that \((\mathcal{C}, \alpha)\) is a 2-position serious equilibrium under \( V \).

Our result then follows since conditions (1) and (2) in Claim 1.1 are similar for any \( V \).\(^{47}\)

**Proof of Proposition 2.** Recall that when the election is held under PV all serious equilibria are 1- or 2-position equilibria. Moreover, the 1-position equilibrium set is strongly equivalent under any two MVs. Thus, we need consider only the 2-position serious equilibria.

Let \( x_L \) and \( x_R \) denote the two policies, with \( x_L < x_R \), and call symmetric an equilibrium where \( v^L = v^R \). We shall prove the result by showing that the 2-position symmetric serious equilibrium set is strongly equivalent under any two MVs. This is done via the following three claims. The first one characterizes the number of candidates running for election. We have:

**Claim 2.1.** Let the election be held under \( V \) an arbitrary MV, and suppose a 2-position symmetric serious equilibrium. If \( q = 1 \), then \( c_L = c_R = 1 \). Otherwise, \( 2 \leq c_L = c_R \leq q \) with \( c_L \neq q - 1 \).

**Proof of Claim 2.1.** Take \((\mathcal{C}, \alpha)\) a 2-position serious equilibrium under \( V \) with \( v^L = v^R \). From Lemma 3 in Dellis (2006) we know that there cannot be more than \( q \) serious contenders on each platform. The result is then trivial for \( q = 1 \). Thus, suppose from now on that \( q \geq 2 \).

To see that \( c_L = c_R \), let \( c_L \neq c_R \). W.l.o.g. let \( (c_R + 1) \leq c_L \leq q \). Since candidates are running on only two platforms, voting must be sincere (by weak undominance). Letting \( s_k = 1 \) if \( k \leq \min\{q, c - 1\} \) and 0 otherwise, we thus have

\[
\begin{align*}
\pi_i(\mathcal{C}, \alpha) &\geq \frac{N-M}{2} + \frac{1}{c_L} \sum_{k=c_L+1}^{c} s_k \quad \text{for some } i \in \mathcal{C}_L; \text{ and} \\
\pi_j(\mathcal{C}, \alpha) &\leq \frac{N-M}{2} + \frac{1}{c_R} \sum_{k=c_R+1}^{c} s_k \quad \text{for some } j \in \mathcal{C}_R
\end{align*}
\]

where \( \pi_i(\mathcal{C}, \alpha) \) is candidate \( i \)'s vote total and \( \mathcal{C}_h \) the set of candidates running at \( x_h \) \((h = L, R)\). But \( i \) and \( j \) two serious contenders implies that \( \pi_i(\mathcal{C}, \alpha) = \pi_j(\mathcal{C}, \alpha) \). It must then be that \( \sum_{k=c_L+1}^{c} s_k \geq c_R \), a contradiction. Hence \( c_L = c_R \).

To see that \( c_L = c_R \geq 2 \), assume by way of contradiction that \( c_L = c_R = 1 \). Since \((\mathcal{C}, \alpha)\) is an equilibrium it must be that neither candidate is better off not running and no other citizen is willing to stand for election. The former implies \(-\frac{c_{R}}{M} \geq \delta \), while the latter requires \(-\frac{c_{L}}{M} < \delta \), a contradiction.

Finally, suppose by way of contradiction that \( c_L = c_R = (q - 1) \). First pick \( k \in (N \setminus \mathcal{C}) \) with \( x_k = x_L \) (where \( N \setminus \mathcal{C} \) denotes the set-theoretic difference between

\(^{47}\)Note that under PV the equilibrium set is the same whether vote cumulation or vote truncation is allowed or not. Hence, Claim 1.1 provides a characterization of the 2-position serious equilibrium set under PV.
$\mathcal{N}$ and $\mathcal{C}$, and construct $\tilde{\mathcal{C}} \equiv (\mathcal{C} \cup \{k\})$. Given $\tilde{\mathcal{C}}$ there exists a unique weakly undominated voting strategy for any $\ell \in \mathcal{N}_L$, which is such that $\alpha_i^\ell (\tilde{\mathcal{C}}) = 1$ for all $i \in \tilde{\mathcal{C}}_L$ and $\alpha_j^\ell (\tilde{\mathcal{C}}) = 0$ for all $j \in \tilde{\mathcal{C}}_R$. At the same time, for any $\ell \in \mathcal{N}_R$ we have $\alpha_i^\ell (\tilde{\mathcal{C}}) = 1$ for some candidate $i \in \tilde{\mathcal{C}}_L$. Finally $G^\ell (\tilde{\mathcal{C}}) = L^\ell (\tilde{\mathcal{C}})$ for all $\ell \in \mathcal{M}$, and thus $\alpha^\ell (\tilde{\mathcal{C}}) = (0, ..., 0)$. Hence, $\pi_j (\tilde{\mathcal{C}}, \alpha) = (\frac{N-M}{2})$ for all $j \in \mathcal{C}_R$, while $\pi_i (\tilde{\mathcal{C}}, \alpha) \geq (1 + \frac{1}{q}) (\frac{N-M}{2})$ for some $i \in \tilde{\mathcal{C}}_L$. Thus, in any voting equilibrium following citizen $k$’s entry in the race we have $W (\tilde{\mathcal{C}}, \alpha) \subseteq \tilde{\mathcal{C}}_L$, and thus a necessary condition for $\mathcal{C}$ to be an equilibrium set of candidates is that $-\frac{v^L_h}{2} < \delta$ - i.e. citizen $k$ does not want to enter the race. Now pick $i \in \mathcal{C}_L$, and define $\bar{\mathcal{C}} \equiv (\mathcal{C} \setminus \{i\})$. Given $\bar{\mathcal{C}}$ we have $U^i (\bar{\mathcal{C}}, \alpha) \geq v^R_k$ in any voting equilibrium. Moreover $(\mathcal{C}, \alpha)$ an equilibrium implies that citizen $i$ is better off standing for election; that is, $U^i (\mathcal{C}, \alpha) \geq U^i (\bar{\mathcal{C}}, \alpha)$. Hence $-\frac{v^R_k}{2} \geq \delta$, a contradiction since $v^R_k = v^L_k$ and $-\frac{v^L_h}{2} < \delta$. ||

The next claim considers the issue of entry at the non-winning positions.

**Claim 2.2.** Let $V$ be an arbitrary MV, and suppose that candidates are standing on two platforms - say, $x_L$ and $x_R$ - which are such that $v^L_k = v^R_k$. If the conditions stated in Claim 2.1 are satisfied and $W (\mathcal{C}, \alpha) = \mathcal{C}$, then there exists an equilibrium profile of voting strategies $\alpha (\cdot)$ such that $U^i (\mathcal{C} \cup \{i\}, \alpha) < U^i (\bar{\mathcal{C}}, \alpha)$ for all citizen $i$ with $x_i \notin \{x_L, x_R\}$.

**Proof of Claim 2.2.** Pick a citizen $k$ with $x_k \notin \{x_L, x_R\}$ - if such a citizen does not exist, then we are done -. and define $\bar{\mathcal{C}} \equiv (\mathcal{C} \cup \{k\})$. W.l.o.g. suppose $x_k < x_L$. Note that $G^k (\bar{\mathcal{C}}) \neq L^k (\bar{\mathcal{C}})$ for all citizen $\ell$, and thus by weak undominance $\alpha^k (\bar{\mathcal{C}}) \neq (0, ..., 0)$. We are going to prove the result by constructing $\alpha (\bar{\mathcal{C}})$.

Let’s first consider the case where $c > q$. Suppose first that $c_L = c_R = q$. Then for all $\ell \in \mathcal{N}_L$ let $\alpha_i^\ell (\bar{\mathcal{C}}) = 1$ for all $i \in \mathcal{C}_L$, and for all $\ell \in \mathcal{N}$ with $x_\ell \geq m$ let $\alpha_j^\ell (\bar{\mathcal{C}}) = 1$ for all $j \in \mathcal{C}_R$. Hence $\alpha (\bar{\mathcal{C}})$ is a profile of weakly undominated voting strategies, and $W (\bar{\mathcal{C}}, \alpha) = \mathcal{C}_R$. Suppose instead that $c_L = c_R \leq (q - 2)$. Pick $S \subset \mathcal{C}_R$ with $\#S = |q - (c_L + 1)|$. Construct $\alpha (\bar{\mathcal{C}})$ as follows: (1) for all $\ell \in \mathcal{N}_L$, let $\alpha_i^\ell (\bar{\mathcal{C}}) = 1$ for all $i \in (\mathcal{C}_L \cup S \cup \{k\})$; (2) for all $\ell \in \mathcal{N}$ with $x_\ell \geq m$, let $\alpha_j^\ell (\bar{\mathcal{C}}) = 1$ for all $j \in \mathcal{C}_R$; (3) $\alpha_h^\ell (\bar{\mathcal{C}}) = 0$ for all $\ell \in \mathcal{M}$; and (4) for any $i \in (\mathcal{C}_L \cup \{k\})$, let $\alpha_h^\ell (\bar{\mathcal{C}}) = \alpha_i^\ell (\bar{\mathcal{C}}) = 0$ for some $h, \ell \in \mathcal{N}$, $h \neq \ell$, $x_h, x_\ell \geq m$. Note that the latter is possible since $(c + 1) \geq (q + 2)$, and thus that there are at least two candidates for whom a citizen cannot cast a vote. Hence $\pi_j (\bar{\mathcal{C}}, \alpha) = N > (N - 2) \geq \pi_h (\bar{\mathcal{C}}, \alpha)$ for all $j \in S$ and $h \in (\bar{\mathcal{C}} \setminus S)$, and thus $W (\bar{\mathcal{C}}, \alpha) = S$.

Let’s now consider the case where $c \leq q$. Hence, each citizen votes for all candidates but one. Pick $j \in \mathcal{C}_R$ arbitrarily, and define $\bar{S} \equiv (\mathcal{C}_R \setminus \{j\})$. Construct $\alpha (\bar{\mathcal{C}})$ as follows: (1) for all $\ell \in \mathcal{N}_L$ and any $h \in \bar{\mathcal{C}}$, $h \neq j$, let $\alpha_h^\ell (\bar{\mathcal{C}}) = 1$; and (2) for all $\ell \in \mathcal{M}$ and any $h \in \bar{\mathcal{C}}$, $h \neq k$, let $\alpha_h^\ell (\bar{\mathcal{C}}) = 1$. Moreover if $c \geq 6$, then let
\( \alpha (\bar{C}) \) be such that for every \( i \in C_L \) there exists \( \ell \in N_R \) with \( \alpha_i^L (\bar{C}) = 0 \). Note that this is possible since \( c_L = c_R = \frac{N-M}{2} \). Instead if \( c = 4 \), then let \( \alpha (\bar{C}) \) be such that there exists \( \ell \in N_R \) such that \( \alpha_i^L (\bar{C}) = 0 \), and for any \( i \in C_L \) there exist \( h, \ell \in N_R \), \( h \neq \ell \), such that \( \alpha_i^L (\bar{C}) = \alpha_i^L (\bar{C}) = 0 \). Note that the latter is possible since \( \frac{N-M}{2} \geq 6 \). Hence \( \alpha (\bar{C}) \) is a profile of weakly undominated voting strategies, and \( W (\bar{C}, \alpha) = S \) with no voter willing to deviate.

In all cases, \( W (\bar{C}, \alpha) = S \subset C_R \), and thus \( U^k (\bar{C}, \alpha) < U^k (C, \alpha) \).

The proof for \( x_k \in (x_L, x_R) \) follows closely the above proof, and is therefore omitted here. 

The next claim considers the issue of entry at the winning positions.

**Claim 2.3.** Let \( \mathcal{C} \) be a two-position set of candidates with \( v_L^m = v_R^m \) and \( c_L = c_R = \pi \), where \( \pi = q \) if \( q \leq 3 \) and \( \pi = 2 \) otherwise. Then there exists an equilibrium profile of voting strategies such that \( U^i (\mathcal{N} \setminus \{i\}, \alpha) < U^i (C, \alpha) \) for all \( i \in (\mathcal{N} \setminus \mathcal{C}) \) with \( x_i \in \{x_L, x_R\} \).

**Proof of Claim 2.3.** Pick \( k \in (\mathcal{N} \setminus \mathcal{C}) \) with \( x_k \in \{x_L, x_R\} \), and let \( \bar{C} \equiv (\mathcal{N} \setminus \{k\}) \). W.l.o.g. suppose \( x_k = x_L \). Also, note that for all \( \ell \in \mathcal{M} \), \( \alpha^L (\mathcal{C}) = \alpha^L (\bar{C}) = (0, ..., 0) \). We are going to construct \( \alpha (\mathcal{C}) \) and \( \alpha (\bar{C}) \) such that \( U^k (\bar{C}, \alpha) < U^k (C, \alpha) \).

First suppose \( q \leq 3 \). By assumption, \( c_L = c_R = q \). Construct \( \alpha \) as follows: (1) for all \( \ell \in N_L \), let \( \alpha^L_i (\mathcal{C}) = \alpha^L_i (\bar{C}) = 1 \) for any \( i \in C_L \); and (2) for all \( \ell \in N_R \), let \( \alpha^L_j (\mathcal{C}) = \alpha^L_j (\bar{C}) = 1 \) for any \( j \in C_R \). Note that \( \alpha (\mathcal{C}) \) and \( \alpha (\bar{C}) \) are two profiles of weakly undominated voting strategies, and that no citizen has an incentive to deviate. Hence \( W (\mathcal{C}, \alpha) = W (\bar{C}, \alpha) = \mathcal{C} \), and \( U^k (\bar{C}, \alpha) < U^k (C, \alpha) \).

Now suppose \( q \geq 4 \). By assumption, \( c_L = c_R = 2 \). Hence \( c \leq q \), and under both \( \mathcal{C} \) and \( \bar{C} \) each non-median citizen votes for all candidates but one. Now, construct \( \alpha (\mathcal{C}) \) as follows: (1) for all \( \ell \in N_L \), let \( \alpha^L_i (\mathcal{C}) = 1 \) for all \( i \in C_L \) \((N_R, \text{ resp.})\), let \( \alpha^L_i (\mathcal{C}) = 1 \) for all \( i \in C_L \) \((N_R, \text{ resp.})\); and (2) \( \alpha (\bar{C}) \) is such that for any \( k \in (\mathcal{C} \setminus \mathcal{C}) \), \( \sum_{\ell \in N_L} \alpha^L_k (\mathcal{C}) = \frac{N-M}{2} \) \((h = L, R)\).

Hence \( \alpha (\mathcal{C}) \) is an equilibrium profile of weakly undominated voting strategies, and \( W (\mathcal{C}, \alpha) = \mathcal{C} \). Also, construct \( \alpha (\bar{C}) \) as follows: (1) pick \( j \in C_R \), and for all \( \ell \in N_L \), let \( \alpha^L_h (\bar{C}) = 1 \) for any \( h \in (\bar{C} \cup \{j\}) \); (2) for all \( \ell \in N_R \), let \( \alpha^L_h (\bar{C}) = 1 \) for all \( h \in C_R \); and (3) let \( \alpha (\bar{C}) \) be such that for every \( i \in \bar{C}_L \) there exist \( h, \ell \in N_R \), \( h \neq \ell \), with \( \alpha^L_h (\bar{C}) = \alpha^L_\ell (\bar{C}) = 0 \). The latter is possible since \( c_L = 3 \) and \( \left( \frac{N-M}{2} \right) \geq 6 \). Note that \( \alpha (\bar{C}) \) is an equilibrium profile of voting strategies, and \( W (\bar{C}, \alpha) = \{j\} \).

Hence \( U^k (\bar{C}, \alpha) < U^k (C, \alpha) \). 

We are now ready to prove our result. Take \((\mathcal{C}, \alpha)\) a 2-position serious equilibrium under PV. We know from Besley and Coate (1997) - or Claim 1.1 - that \( \mathcal{C} = \{i, j\} \) with \( x_i = x_L \) and \( x_j = x_R \) such that \( v_L^m = v_R^m \). Moreover, it must be
that none of the candidates would be better off not running for election; that is, 
\[ -\frac{v^j_i}{\bar{v}} \geq \delta \] and \[ -\frac{v^j_k}{\bar{v}} \geq \delta. \]

Take \( \bar{V} \) an arbitrary MV, and let \( \bar{\eta} \in \mathbb{N} \) denote the maximum number of votes a citizen is ever allowed to cast under \( \bar{V} \). If \( \bar{\eta} = 1 \), then the result is trivially established. So assume from now on that \( \bar{\eta} \geq 2 \). Construct \( \bar{C} \) such that \( x_k \in \{x_L, x_R\} \) for all \( k \in \bar{C} \), with \( \bar{\tau}_h = \bar{\eta} \) if \( \bar{\eta} \in \{2, 3\} \) and \( \bar{\tau}_h = 2 \) otherwise (for \( h = L, R \)). Also, let \( \bar{\pi} \) be a voting profile under \( \bar{V} \). For \( \bar{\eta} \in \{2, 3\} \) there is a unique equilibrium profile of voting strategies, with \( W(\bar{C}, \bar{\pi}) = \bar{C} \). For \( \bar{\eta} \geq 4 \), construct \( \bar{\pi}(\bar{C}) \) as follows: (1) for all \( \ell \in \mathcal{N}_L \), let \( \bar{\pi}^\ell_i (\bar{C}) = 1 \) for all \( i \in \bar{C}_L \); (2) for all \( \ell \in \mathcal{N}_R \), let \( \bar{\pi}^\ell_j (\bar{C}) = 1 \) for all \( j \in \bar{C}_R \); (3) for all \( \ell \in \mathcal{M} \), let \( \bar{\pi}^\ell (\bar{C}) = (0, \ldots, 0) \); and (4) let \( \bar{\pi}(\bar{C}) \) be such that for any \( k \in (\bar{C}\setminus\bar{C}_h) \), \( \sum_{\ell \in \mathcal{N}_h} \bar{\pi}^\ell_k (\bar{C}) = \frac{\bar{\eta} - h}{\bar{\eta} - 1} \) (for \( h = L, R \)).

Hence \( W(\bar{C}, \bar{\pi}) = \bar{C} \) for any \( \bar{\eta} \). Moreover we know from Claims 2.2 and 2.3 that there exists an equilibrium profile of voting strategies such that no other citizen is willing to enter the race. Also, none of the candidates in \( \bar{C} \) would be better off not running for office since \( -\frac{v^j_i}{\bar{v}} \geq \delta \) and \( -\frac{v^j_k}{\bar{v}} \geq \delta \). Hence, \( (\bar{C}, \bar{\pi}) \) is an equilibrium under \( \bar{V} \) which is strongly equivalent to \( (C, \alpha) \). \( \blacksquare \)

**Proof of Proposition 3.** As mentioned in the text we only need consider 2-position serious equilibria to prove the result.

Take \((C, \alpha)\) a 2-position serious equilibrium under \( V \) - if such an equilibrium does not exist, then we are done. Pick \( i \) and \( j \) in \( C \), with \( x_i = x_L \) and \( x_j = x_R \), and let \( \bar{C} = \{i, j\} \). Since \( \bar{\eta} = 2 \), voting must be sincere in any equilibrium (by weak undomiance). Also, \((C, \alpha)\) an equilibrium under \( V \) implies that \( -\frac{v^j_i}{\bar{v}} \geq \delta \) and \( -\frac{v^j_k}{\bar{v}} \geq \delta \); that is, neither candidate \( i \), nor candidate \( j \) would be better off not running for election. Now, \( \frac{1}{2} \geq \frac{v^j_i}{\bar{v}} \) (for \( h = L, R \)) and \( \bar{\eta} = 2 \) imply that the same holds true under PV when \( \bar{C} \) is the set of candidates. It remains to show that no other citizen wants to run for election. Pick \( k \in (\mathcal{N}\setminus\bar{C}) \) arbitrarily, and define \( \bar{C} = (\bar{C} \cup \{k\}) \). Construct \( \bar{\pi}(\bar{C}) \) as follows: (1) for all \( \ell \in \mathcal{N}_L \), let \( \bar{\pi}^\ell_i (\bar{C}) = 1 \); (2) for all \( \ell \in \mathcal{M} \), let \( \bar{\pi}^\ell (\bar{C}) = (0, \ldots, 0) \); and (3) for all \( \ell \in \mathcal{N}_R \), let \( \bar{\pi}^\ell_j (\bar{C}) = 1 \); (4) for all \( \ell \in \mathcal{N}_R \), let \( \bar{\pi}^\ell_k (\bar{C}) = 1 \).

Hence, \( W(\bar{C}, \bar{\pi}) = \bar{C} \) if \( x_k \notin [x_L, x_R] \) and \( \{h\} \) otherwise. Thus, \( U^k(\bar{C}, \bar{\pi}) < U^k(\bar{C}, \bar{\pi}) \), and \((\bar{C}, \bar{\pi})\) is a 2-position serious equilibrium under PV with \((x_L, x_R)\) as the policy outcome. Since we took \((C, \alpha)\) arbitrarily this must be true for any 2-position serious equilibrium under \( V \). Hence the subset part of the result.

For any \( x_L \) and \( x_R \in X \), with \( v^m_L = v^m_R \) and \( n_L, n_R \geq 1 \), define \( q(x_L, x_R) \) as the integer that satisfies
\[
\frac{1}{2} \left( 1 - \frac{v^m_R}{2 \bar{v}} \right) < q(x_L, x_R) \leq \frac{1}{2} \left( 1 - \frac{v^m_L}{2 \bar{v}} \right)
\]
and let \( \bar{\eta} \) be the largest \( q(x_L, x_R) \). Hence, \( \bar{\eta} \) is the largest number of candidates who would be willing to run on the same platform in a 2-position serious equilibrium. Suppose \( q > \bar{\eta} \) and \( u^\ell(x) = u(|x_L - x|) \) for all citizen \( \ell \).
Pick \((C, \alpha)\) a 2-position serious equilibrium under \(V\), with \({x_L, x_R}\) as the policy outcome - if such an equilibrium does not exist, then we are done. Take \((\bar{C}, \bar{\alpha})\) a 2-position serious equilibrium under PV with \({\bar{x}_L, \bar{x}_R}\) as the policy outcome, \([\bar{x}_L, \bar{x}_R] \subseteq [x_L, x_R]\). We are going to show that there exists a strongly equivalent equilibrium under \(V\). We know from Claim 1.1 that \(\pi_L = \bar{\pi}_R = 1\). Construct \(\bar{C}\) such that: (1) \(x_i \in \{\bar{x}_L, \bar{x}_R\}\) for all \(i \in \bar{C}\); (2) \(\bar{C} \subseteq \bar{C}\); (3) \(\frac{\bar{\pi}_R}{\bar{\pi}_L + \bar{\pi}_R} \geq \delta\) for all \(i \in \bar{C}_L\), where \(\bar{\pi}_L = \bar{\pi}_R\), and a similar condition holds for any \(j \in \bar{C}_R\); and (4) there does not exist \(i \in (\bar{V} \setminus \bar{C})\) with \(x_i = \bar{x}_L\) and \(\frac{\bar{\pi}_R}{\bar{\pi}_L + \bar{\pi}_R} \geq \delta\), and a similar condition for \(j \in (\bar{V} \setminus \bar{C})\) with \(x_j = \bar{x}_R\). Note that \(\bar{\pi}_L = \bar{\pi}_R < q\), the equality by symmetry of preferences and the inequality by \(q > \bar{\pi}\). Also, note that for every citizen \(\ell\) either \(G^\ell \left(\bar{C}\right) = L^\ell \left(\bar{C}\right)\), or \(L^\ell \left(\bar{C}\right) \neq G^\ell \left(\bar{C}\right)\) and \(\bar{C} = \left(L^\ell \left(\bar{C}\right) \lor G^\ell \left(\bar{C}\right)\right)\). The same is true for every set of candidates \((\bar{C} \setminus \{k\}), k \in \bar{C}\). Hence voting must be sincere (by weak undomination). Now condition (3) guarantees that neither candidate in \(\bar{C}\) would be better off not running for election under \(V\). It remains to show that there exists a profile of voting strategies such that no other citizen wants to run for election. Pick \(k \in (\bar{V} \setminus \bar{C})\) arbitrarily. Condition (4) guarantees that citizen \(k\) does not want to run for election if \(x_k \in \{\bar{x}_L, \bar{x}_R\}\). Suppose instead that \(x_k < \bar{x}_L\). Construct \(\alpha(\bar{C} \cup \{k\})\) as follows:

\[
\begin{align*}
\alpha^L_k(\bar{C} \cup \{k\}) &= 1 \quad \text{for all } \ell \in \bar{N}_L \text{ and } i \in \bar{C}_L; \\
\alpha^k_k(\bar{C} \cup \{k\}) &= 1 \quad \text{for all } \ell \in \bar{N} \text{ with } k \in G^\ell \left(\bar{C} \cup \{k\}\right); \\
\alpha_j^k(\bar{C} \cup \{k\}) &= 1 \quad \text{for all } \ell \in (\bar{M} \cup \bar{N}_R) \text{ and } j \in \bar{C}_R
\end{align*}
\]

and \(\alpha^L_k(\bar{C} \cup \{k\}) = 0\) otherwise. Hence \(W(\bar{C} \cup \{k\}, \alpha) = \bar{\pi}_R\), and \(U^k(\bar{C} \cup \{k\}, \alpha) < U^k(\bar{C}, \alpha)\). Construct a similar profile of voting strategies for \(x_k > \bar{x}_R\). Finally, suppose \(x_k \in (\bar{x}_L, \bar{x}_R)\). Given that \([\bar{x}_L, \bar{x}_R] \subseteq [x_L, x_R]\) and \((C, \alpha)\) a 2-position serious equilibrium under \(V\), it must be that \(-\frac{v^L_k + v^R_k}{2} < \delta\) and/or there exists \(\alpha\) such that \(\pi_k(C \cup \{k\}, \alpha) = \gamma^k < \left(\frac{N - M}{2}\right) - 1\), where \(\gamma^k \equiv \#\{\ell \in \bar{N} : k \in G^\ell (C \cup \{k\})\}\). Since \(x_k > \bar{x}_R > x_L\) and \(v^\ell(.)\) a strictly concave utility function we have that \(\bar{v}^L_k > v^k_k\). Similarly, \(\bar{v}^k_R > v^k_k\). Hence \(-\frac{\bar{v}^L_k + \bar{v}^k_R}{2} < -\frac{v^L_k + v^R_k}{2}\). Also, \(\bar{\gamma}^k \leq \gamma^k\), where \(\bar{\gamma}^k \equiv \#\{\ell \in \bar{N} : k \in G^\ell (\bar{C} \cup \{k\})\}\). To see this, take a citizen \(\ell\) such that \(k \in G^\ell (\bar{C} \cup \{k\})\). Then \(x_{\ell} \in (\bar{V} \setminus \bar{x}_R), \) and \(\bar{v}^h_k > v^h_k\) (for \(h = L, R\)). It follows that \(\text{max}\{\bar{v}^L_k, \bar{v}^R_k\} > \text{max}\{v^L_k, v^R_k\}\). But \(k \in G^\ell (\bar{C} \cup \{k\})\) implies that \(v^L_k \geq \text{max}\{\bar{v}^L_k, \bar{v}^R_k\}\), and thus \(v^L_k > \text{max}\{v^L_k, v^R_k\}\). Hence \(G^\ell (C \cup \{k\}) = \{k\}\). Since we took \(\ell\) arbitrarily it must be true for any citizen \(\ell\) for whom \(k \in G^\ell (\bar{C} \cup \{k\})\), and thus \(\bar{\gamma}^k \leq \gamma^k\). It follows that \(-\frac{\bar{v}^L_k + \bar{v}^R_k}{2} < \delta\) and/or \(\bar{\gamma}^k < \left(\frac{N - M}{2}\right) - 1\), and thus there exists \(\alpha\) such that \(U^k(\bar{C} \cup \{k\}, \alpha) < U^k(C \cup \{k\}, \alpha)\). Since \(U^k(C \cup \{k\}, \alpha) \leq U^k(C, \alpha) - (C, \alpha)\) an equilibrium implies that citizen \(k\) does not want to enter the race when the set of candidates is \(\bar{C}\) and \(U^k(\bar{C}, \alpha) < U^k(\bar{C}, \alpha)\) - since \(\bar{v}^h_k > v^h_k\) (for \(h = L, R\)) while \(c_L = c_R\) and \(\bar{c}_L = \bar{c}_R\), the latter by.
Finally, pick $\alpha$ an equilibrium profile of voting strategies for any other set of candidates. Hence $(\vec{C}, \alpha)$ is a 2-position serious equilibrium under $V$ with $\{\vec{x}_L, \vec{x}_R\}$ as the policy outcome. And since $\vec{c}_L = \vec{c}_R$, it is strongly equivalent to $(\vec{C}, \pi)$. ■

Proof of Proposition 4. As noted in the text, we only need consider the 2-position serious equilibrium sets to prove the result. We distinguish two cases.

**Case 1:** Vote truncation is not allowed. Before proving the result, we need to characterize the 2-position serious equilibrium sets under the different OVs. First, it must be that $C = \{L, R\}$ with $x_L \neq x_R$ and $v^u_L = v^u_R$. Moreover, equilibrium voting strategies must be sincere. Also, it must be that: (1) neither of the two candidates would be better off not running for election; and (2) no other citizen would be better off entering the race. Necessary and sufficient conditions for the former to hold are $-\frac{v^u_L}{2} \geq \delta$ and $-\frac{v^u_R}{2} \geq \delta$. The necessary and sufficient conditions for the latter to hold are given in the next three claims. The first one considers the citizens whose ideal policies are $x_L$ and $x_R$.

**Claim 4.1.** Let the election be held under an OV. Then there exists an equilibrium profile of voting strategies such that $U^k(C \cup \{k\}, \alpha) < U^k(C, \alpha)$ for any $k \in (N \setminus C)$ with $x_k \in \{x_L, x_R\}$ if, and only if, $s \leq \frac{1}{2} \left( \frac{N-M-2}{N-M-2} \right)$.

**Proof of Claim 4.1.** Take $k \in (N \setminus C)$ with $x_k \in \{x_L, x_R\}$ - w.l.o.g. suppose $x_k = x_L$ - and let $\vec{C} \equiv \{k, L, R\}$.

(Necessity) Assume by way of contradiction that $s > \frac{1}{2} \left( \frac{N-M-2}{N-M-2} \right)$ and $U^k(C, \alpha) < U^k(C \cup \{k\}, \alpha)$ for some equilibrium profile of voting strategies. First note that $-\frac{v^u_L}{2} \geq \delta$ - i.e. citizen $L$ is willing to stand for election - and $v^u_R = v^u_L$ - since $x_k = x_L$ and $u^\ell(x) = u(x_L - x)$ for all citizen $\ell$ - imply that $U^k(C, \alpha) < U^k(C \cup \{k\}, \alpha)$ only if $R \in W(\vec{C}, \alpha)$. Note also that $G^L(\vec{C}) = \{k, L\}$ (resp.) and $L^k(\vec{C}) = \{R\}$ (resp.) for all $\ell \in N_L$ (resp.), and $L^k(\vec{C}) = G^L(\vec{C})$ for all $\ell \in M$. Hence, in any voting equilibrium the citizens at the median abstain, while the others vote sincerely. This means that $\pi_R(\vec{C}, \alpha) = \frac{N-M}{N-M-2}$ and $\pi_i(\vec{C}, \alpha) \geq \left( \frac{N-M}{N-M-2} \right) \left( \frac{1}{2} + s \right)$ for some $i \in \{k, L\}$. Since there cannot be a voting equilibrium where $W(\vec{C}, \alpha) = \vec{C}$ (see Lemma 3 in Dellis (2006)) candidate $R$ is in the winning set only if either $W(\vec{C}, \alpha) = \{R\}$ or $W(\vec{C}, \alpha) = \{h, R\}$ for some $h \in \{k, L\}$. For the former to hold true it must be that $\pi_R(\vec{C}, \alpha) > \left[ \pi_i(\vec{C}, \alpha) + (1-s) \right]$ for all $i \in \{k, L\}$, which cannot be true if $s \geq \frac{1}{2} \left( \frac{N-M-4}{N-M-2} \right)$. For $W(\vec{C}, \alpha) = \{h, R\}$ it must be that for $i \in \{k, L\}$, $i \neq h$, $\alpha'_i(\vec{C}) = s$ for all $\ell \in (N \setminus M)$ - otherwise some voters would be better off permuting their ranking of the two left candidates - and $\pi_R(\vec{C}, \alpha) = \pi_R(\vec{C}, \alpha) = \frac{N-M}{N-M-2}$ and $\pi_i(\vec{C}, \alpha) = (N - M) s$. But $s > \frac{1}{2} \left( \frac{N-M-2}{N-M-2} \right)$ implies that $\pi_i(\vec{C}, \alpha) + (1-s) > \pi_R(\vec{C}, \alpha)$, and a leftist would then be better off permuting his ranking of the two left candidates. Hence, in any voting equilibrium $R \notin W(\vec{C}, \alpha)$, and $U^k(C \cup \{k\}, \alpha) = U^k(C, \alpha)$ a contradiction.
(Sufficiency) Construct $\alpha \left( \tilde{c} \right)$ as follows

$$\alpha^\ell \left( \tilde{c} \right) = \begin{cases} (s, 1, 0) & \text{for all } \ell \in \mathcal{N}_L \\ (s, 0, 1) & \text{for all } \ell \in \mathcal{N}_R \\ (0, 0, 0) & \text{for all } \ell \in \mathcal{M} \end{cases}$$

, where $\alpha^\ell \left( \tilde{c} \right) \equiv \left( \alpha_k^\ell, \alpha_L^\ell, \alpha_R^\ell \right)$. Hence, $W \left( \tilde{c}, \alpha \right) = W \left( \mathcal{C}, \alpha \right)$ and neither citizen is willing to deviate from his voting strategy. Hence, $U^k \left( \tilde{c}, \alpha \right) < U^k \left( \mathcal{C}, \alpha \right)$. ||

The next claim considers the entry by extremists.

Claim 4.2. Let the election be held under $V$, an arbitrary OV with $s \leq \frac{1}{2} \left( \frac{N - M - 2}{N - 3M - 2} \right)$. Then for any citizen $k$ with $x_k \notin [x_L, x_R]$, there exists an equilibrium profile of voting strategies such that $U^k \left( \mathcal{C} \cup \{k\}, \alpha \right) < U^k \left( \mathcal{C}, \alpha \right)$.

Proof of Claim 4.2. Pick a citizen $k$ with $x_k \notin [x_L, x_R]$ - if there is no such citizen, then we are done -, and let $\tilde{c} \equiv \{k, L, R\}$. W.l.o.g. suppose $x_k < x_L$. Construct $\alpha \left( \tilde{c} \right)$ as follows

$$\alpha^\ell \left( \tilde{c} \right) = \begin{cases} (s, 1, 0) & \text{for all } \ell \in \mathcal{N}_L, \ell \neq k \\ (1, s, 0) & \text{for } \ell = k \\ (0, s, 1) & \text{for all } \ell \in \mathcal{M} \\ (s, 0, 1) & \text{for all } \ell \in \mathcal{N}_R \end{cases}$$

, where $\alpha^\ell \left( \tilde{c} \right) \equiv \left( \alpha_k^\ell, \alpha_L^\ell, \alpha_R^\ell \right)$. We thus have $W \left( \tilde{c}, \alpha \right) = \{k\}$, and no voter is willing to deviate. Hence, $\alpha \left( \tilde{c} \right)$ is an equilibrium profile of voting strategies with $U^k \left( \tilde{c}, \alpha \right) < U^k \left( \mathcal{C}, \alpha \right)$. ||

It remains to consider the entry by moderates. For that purpose, it is convenient to introduce some extra notation. For a candidate $k$ define $\Gamma^k \left( \mathcal{C} \right) \equiv \{\ell \in \mathcal{N} : k \in G^\ell \left( \mathcal{C} \right)\}$ the set of citizens for whom $k$ is a most-preferred candidate, and $\gamma^k \left( \mathcal{C} \right) \equiv \# \Gamma^k \left( \mathcal{C} \right)$ the number of such citizens. Similarly, define $\Gamma^h_k \left( \mathcal{C} \right) \equiv \{\ell \in \mathcal{N}_h : k \in G^\ell \left( \mathcal{C} \right)\}$ and $\gamma^h_k \left( \mathcal{C} \right) \equiv \# \Gamma^h_k \left( \mathcal{C} \right)$, for $h = L, R$.

The next claim provides necessary and sufficient conditions under which for any potential moderate candidate there exists an equilibrium profile of voting strategies such that he does not want to enter the race.

Claim 4.3. Let $\mathcal{C} = \{L, R\}$ with $x_L < x_R$ and $v^m_L = v^m_R$. Suppose the election is held under $V$, an arbitrary OV with $s > 0$ and $s \leq \frac{1}{2} \left( \frac{N - M - 2}{N - 3M - 2} \right)$. Let $k$ be a citizen with $x_k \in (x_L, m]$. There exists an equilibrium profile of voting strategies $\alpha$ such that $U^k \left( \mathcal{C} \cup \{k\}, \alpha \right) < U^k \left( \mathcal{C}, \alpha \right)$:

1. if $M$ is even and $s \leq \frac{N - 2M - 2}{2N - 3M - 2}$, with the latter inequality strict if there exists a citizen $\ell$ with $x_\ell \neq m$ and $G^\ell \left( \mathcal{C} \cup \{\ell\} \right) = \{k\}$;
2. if $\left( \gamma^L_k - \gamma^R_k \right) \geq (M - 1)$;

48 In order to simplify notation I will omit $\mathcal{C}$ whenever it is possible to do so without causing a confusion.
49 While Claim 4.3 does not consider PV (i.e. $s = 0$) it is worth mentioning that $M \leq \frac{N - 4}{2}$ is a sufficient condition in that case.
(3) and otherwise only if at least one of the following conditions hold: 

\[(i) \quad \left( \frac{\alpha - \alpha_k}{2} \right) < \delta; \quad \text{or} \quad (ii) \quad \frac{N-3M-2}{2} > s_R \quad \text{and} \quad \left( \frac{s - s_k}{2} \right) < \delta; \quad \text{or} \quad (iii) \quad \frac{N-3M-2}{2} > s_L, \quad \text{where} \quad s_R \equiv \left( 2\pi_h + \pi_h - \frac{N+M-2}{2} \right) s, \quad \text{for} \ h, i \in \{ L, R \}, \ h \neq i. \quad \text{Moreover condition} \ (i) \ \text{is sufficient.} \]

**Proof of Claim 4.3.** Pick a citizen \( k \) with \( x_k \in (x_L, m] \). Define \( C \equiv \overline{C} \cup \{ k \} \).

Let \( \alpha^\ell (C) \equiv (\alpha^\ell_L, \alpha^\ell_R) \) denote citizen \( \ell \)'s voting strategy.

(1) Suppose that \( M \) is even and \( s \leq \frac{N-3M-2}{2N-3M-2} \), with a strict inequality if there exists a citizen \( \ell \) with \( G^\ell (C) = \{ k \} \) and \( x_\ell \neq m \). Construct \( \alpha (C) \) as follows:

\[
\alpha^\ell (C) = \begin{cases} 
(1, s, 0) & \text{for all} \ \ell \in N_L \\
(0, s, 1) & \text{for all} \ \ell \in N_R 
\end{cases}
\]

and \( \alpha^\ell (C) \in \{(s,0), (0,1)\} \) for all \( \ell \in M \) with \( \sum_{\ell \in M} \alpha^\ell_h (C) = \frac{M}{2} s \), for \( h = L, R \). Hence \( W(C, \alpha) = \overline{C} \), with \( \alpha (C) \) an equilibrium profile of voting strategies, and \( U^k (C, \alpha) < U^k (\overline{C}, \alpha) \).

Suppose from now on that at least one of the conditions in (1) does not hold. It follows that in any voting equilibrium \( \#W(C, \alpha) = 1 \) (the proof is available from the author).

(2) Suppose \( (\gamma_L^k - \gamma_R^k) \geq (M - 1) \). Construct \( \alpha (C) \) as follows:

\[
\alpha^\ell (C) = \begin{cases} 
(1, 0, s) & \text{for all} \ \ell \in \mathcal{N} \text{ with } G^\ell (C) = \{ L \} \\
(1, s, 0) & \text{for all} \ \ell \in \Gamma^k_L \\
(0, 1, s) & \text{for all} \ \ell \in \mathcal{M} \\
(0, s, 1) & \text{for all} \ \ell \in \Gamma^k_R 
\end{cases}
\]

, and let \( \alpha^\ell (C) = (0, s, 1) \) for \( (\gamma_L^k - \gamma_R^k - M + 2) \) of the citizens for whom \( G^\ell (C) = \{ R \} \) and \( (s, 0, 1) \) for the other ones. Note that \( [\pi_R (C, \alpha) - \pi_L (C, \alpha)] = 2 s \) and \( [\pi_R (C, \alpha) - \pi_k (C, \alpha)] > (1 + s) \), the latter following from \( (3\gamma_L^k + 3 - \frac{N+3M}{2}) < M - \frac{N-3M-2}{2N-3M-2} \) (the proof is available from the author). Hence, \( W(C, \alpha) = \{ R \} \) with \( \alpha (C) \) an equilibrium profile of voting strategies, and \( U^k (C, \alpha) < U^k (\overline{C}, \alpha) \) since \( R \in L^k (\overline{C}) \) and \( \delta > 0 \).

(3) Suppose \( (\gamma_L^k - \gamma_R^k) < (M - 1) \). I am first going to prove the necessity part. Construct \( \alpha (C) \) as above but with \( \alpha^\ell (C) = (s, 0, 1) \) for all citizen \( \ell \) for whom \( G^\ell (C) = \{ R \} \). Hence, \( \alpha (C) \) is the profile of weakly undominated voting strategies that maximizes \( [\pi_R (C, \alpha) - \pi_k (C, \alpha)] \). If condition (iii) is not satisfied this difference is lower than \((1 + s)\), and either \( k \) wins outright or citizen \( L \) wants to deviate and cast a ballot \((s, 1, 0)\); that is, \( W(C, \alpha) \in \{ k \} \) in any voting equilibrium. Now, suppose that condition (ii) does not hold either. Either

\footnote{Note that the set of sufficient conditions stated here is not tight. However if neither of those conditions holds, then in any other set of sufficient conditions either condition (b) must hold with an equality, or condition (c) must hold with the second inequality replaced by an equality (the proof is available from the author). In any case the proof of Proposition 4 below would still hold if we include those conditions.}
\((\frac{s^k - s_R^k}{s - s_R}) \geq \delta\), in which case condition \((i)\) does not hold and \(U^k(C, \alpha) \geq U^k(\overline{C}, \alpha)\).

Or \(s_R \geq \frac{N - 3M - 2}{2}\), in which case it cannot be that candidate \(L\) wins outright. To see why, construct \(\alpha(C)\) as above, replacing \(\alpha'(C) = (0,1, s)\) by \((s,1,0)\) for all \(\ell \in \mathcal{M}\). Hence \(\alpha(C)\) is the profile of weakly dominated voting strategies that maximizes \(\pi_L(C, \alpha) - \pi_k(C, \alpha)\). But this difference is smaller than \((1 + s)\) (given the assumption on \(s_R\)), and either \(k\) wins outright or citizen \(R\) wants to deviate and cast a ballot \((0,1,s)\) since then candidate \(k\) would either tie with or defeat candidate \(L\). Hence if neither condition \((ii)\), nor condition \((iii)\) holds, then in any voting equilibrium \(W(C, \alpha) = \{k\}\). And if condition \((i)\) does not hold either, then \(U^k(C, \alpha) \geq U^k(\overline{C}, \alpha)\).

Condition \((i)\) implies that in any voting equilibrium \(U^k(C, \alpha) < U^k(\overline{C}, \alpha)\), and is thus sufficient. Condition \((iii)\) is sufficient as well if the additional conditions stated hold since no \(\ell \in \mathcal{N}_R\) for whom \(G'(\overline{C}) = \{k\}\) wants to deviate from his voting strategy, either because there is no such citizen \(- (a)\), or he is not pivotal for candidate \(k\) \(- (b)\), or candidate \(L\) would be winning \(- (c)\). Similarly for condition \((ii)\) if \((M - 1) > (\gamma_R^k - \gamma_L^k)\), the latter condition for candidate \(L\) to defeat candidate \(R\).

We are now ready to prove Proposition 4 for the case where vote truncation is not allowed. Take \(V\) and \(\overline{V}\) any two OVs with \(s > \pi\). From Claim 4.1 we know that if \(s > \frac{1}{2} \left(\frac{N - M - 3}{N - M - 1}\right)\) the set of 2-position serious equilibria under \(V\) is empty, and the result holds trivially. Suppose from now on that \(s \leq \frac{1}{2} \left(\frac{N - M - 3}{N - M - 1}\right)\).

To show that the set of 2-position serious equilibria under \(V\) is a subset of the set of 2-position serious equilibria under \(\overline{V}\) it is sufficient to show that for any such equilibrium under \(V\) there exists a strongly equivalent one under \(\overline{V}\). Take \((C, \alpha)\) a 2-position serious equilibrium under \(V\), and call \(L\) and \(R\) the two candidates. I am going to construct \((\overline{C}, \overline{\alpha})\) a strongly equivalent serious equilibrium under \(\overline{V}\). Let \(\overline{C} \equiv C\) and \(\overline{\alpha}(\overline{C}) \equiv \alpha(C)\). Since \((C, \alpha)\) is a 2-position serious equilibrium under \(V\) it must be that neither candidate would be better off not running; that is, \(-s^k_L \geq \delta\) and \(-s^k_R \geq \delta\). Pick \(k \in (\mathcal{N}\backslash \overline{C})\) arbitrarily, and let \(\overline{C} \equiv \{L, R, k\}\). We need to construct \(\overline{\alpha}(\overline{C})\) an equilibrium profile of voting strategies such that \(U^k(\overline{C}, \overline{\alpha}) < U^k(\overline{C}, \alpha)\). Since \(\pi < s\) we know from Claims 4.1 and 4.2 that if \(x_k \notin (x_L, x_R)\) such an equilibrium profile of voting strategies exists. Suppose instead that \(x_k \in (x_L, x_R)\). Since \((C, \alpha)\) is a serious equilibrium under \(V\) there must exist an equilibrium profile of voting strategies \(\alpha(\overline{C})\) such that \(U^k(\overline{C}, \alpha) < U^k(\overline{C}, \alpha)\). And given that \(\pi < s\) and \(M \leq \frac{N - 1}{3}\) the same holds true under \(\overline{V}\). Hence, for any \(k \in (\mathcal{N}\backslash \overline{C})\) there exists \(\overline{\alpha}(\overline{C})\) such that \(U^k(\overline{C}, \overline{\alpha}) < U^k(\overline{C}, \alpha)\). Finally, for any other set of candidates take any equilibrium profile of voting strategies. Hence \((\overline{C}, \overline{\alpha})\) is a 2-position serious equilibrium under \(\overline{V}\), which moreover is strongly equivalent to \((C, \alpha)\).

It remains to show that this subset is moderate. From above we know that the set of 2-position serious equilibria under any OV is a subset of the set of 2-position serious equilibria under PV. Hence, to prove our result it is sufficient to show that under any OV the set of 2-position serious equilibria is a moderate subset of the one under PV. Take \(V\) an OV, and suppose there exists \((C, \alpha)\) a 2-position serious equilibrium under \(V\). Note that this condition is necessarily satisfied if condition \((iii)\) does not hold.
equilibrium under $V$ with \( \{x_L, x_R\} \) as the policy outcome. Assume also that there exists \((\mathcal{C}, \pi)\) a 2-position serious equilibrium under PV, with \(\{\hat{x}_L, \hat{x}_R\}\) as the policy outcome, \([\hat{x}_L, \hat{x}_R] \subseteq [x_L, x_R]\). I am going to construct \((\tilde{\mathcal{C}}, \alpha)\) a strongly equivalent serious equilibrium under \(V\). Let \(\tilde{\mathcal{C}} \equiv \mathcal{C}\) and \(\alpha(\tilde{\mathcal{C}}) \equiv \pi(\mathcal{C})\). Since \((\mathcal{C}, \pi)\) is an equilibrium it must be that neither candidate would be better off not running that is, \(-\frac{\hat{v}^L_k}{k} \geq \delta\) and \(-\frac{\hat{v}^R_k}{k} \geq \delta\) where \(\hat{v}^L_k = u(\hat{x}_L - \hat{x}_R)\) and \(\hat{v}^R_k = u(\hat{x}_R - \hat{x}_L)\).

Pick \(k \in (\mathcal{N} \setminus \tilde{\mathcal{C}})\) arbitrarily, and define \(\tilde{\mathcal{C}} \equiv (\tilde{\mathcal{C}} \cup \{k\})\). If \(x_k \notin (\hat{x}_L, \hat{x}_R)\), then we know by Claims 4.1 and 4.2 that there exists \(\alpha(\tilde{\mathcal{C}})\) such that \(U^k(\tilde{\mathcal{C}}, \alpha) < U^k(\mathcal{C}, \alpha)\). Suppose instead that \(x_k \in (\hat{x}_L, \hat{x}_R)\). W.l.o.g. let \(x_k \in (\hat{x}_L, m)\). Since \((\mathcal{C}, \alpha)\) is a 2-position serious equilibrium under \(V\) there must exist an equilibrium profile of voting strategies \(\alpha(\mathcal{C} \cup \{k\})\) such that \(U^k(\mathcal{C} \cup \{k\}, \alpha) < U^k(\mathcal{C}, \alpha)\). It can then be shown that the same holds true under \(\tilde{\mathcal{C}}\) (the proof is available from the author). Hence, for any \(k \in (\mathcal{N} \setminus \tilde{\mathcal{C}})\) there exists an equilibrium profile of voting strategies such that \(U^k(\tilde{\mathcal{C}}, \alpha) < U^k(\mathcal{C}, \alpha)\).

Finally, for any other set of candidates take any equilibrium profile of voting strategies.

Hence \((\tilde{\mathcal{C}}, \alpha)\) is a 2-position serious equilibrium under \(V\), which in addition is strongly equivalent to \((\mathcal{C}, \pi)\).

**Case 2:** Vote truncation is allowed. The characterization of the 2-position serious equilibrium set is still the same, except for the entry by new candidates. The following claim replaces Claims 4.1-4.3.

**Claim 4.4.** Let the election be held under an OV, and suppose that citizens are allowed to truncate their ballot. Then there exists an equilibrium profile of voting strategies such that \(U^k(\mathcal{C} \cup \{k\}, \alpha) < U^k(\mathcal{C}, \alpha)\) for any \(k \in (\mathcal{N} \setminus \mathcal{C})\) with \(x_k \notin (x_L, x_R)\). The same holds true for any citizen \(k\) with \(x_k \in (x_L, x_R)\) if, and only if, at least one of the following two conditions holds: (1) \(\frac{v^{\ell}_k + v^{\ell}_R}{2} < \delta\); or (2) \((\gamma^L_k + \gamma^R_k)s \leq \frac{M - 3M - 2}{2}\).

**Proof of Claim 4.4.** Let \(\mathcal{C} = \{L, R\}\) be a set of candidates with \(x_L < m < x_R\) and \(v^L_R = v^R_L\). Pick \(k \in (\mathcal{N} \setminus \mathcal{C})\) arbitrarily, and define \(\tilde{\mathcal{C}} \equiv \{k, L, R\}\).

First consider the case where \(x_k \notin [x_L, x_R]\) - w.l.o.g. suppose \(x_k < x_L\). Construct \(\alpha(\tilde{\mathcal{C}})\) as follows:

\[
\alpha^\ell(\tilde{\mathcal{C}}) = \begin{cases} 
(s, 1, 0) & \text{for all } \ell \in \mathcal{N}_L \text{ with } k \in G^\ell(\tilde{\mathcal{C}}) \\
(0, 1, 0) & \text{for all } \ell \in \mathcal{N}_L \text{ with } k \notin G^\ell(\tilde{\mathcal{C}}) \\
(0, s, 1) & \text{for all } \ell \in \mathcal{M} \\
(0, 0, 1) & \text{for all } \ell \in \mathcal{N}_R
\end{cases}
\]

, where \(\alpha^\ell(\tilde{\mathcal{C}}) = (\alpha^\ell_L, \alpha^\ell_R)\). Hence, \(W(\tilde{\mathcal{C}}, \alpha) = \{R\}\) with \(\alpha(\tilde{\mathcal{C}})\) an equilibrium profile of voting strategies. This means that \(U^k(\tilde{\mathcal{C}}, \alpha) < U^k(\mathcal{C}, \alpha)\).

Now consider the case where \(x_k \in \{x_L, x_R\}\) - w.l.o.g. suppose \(x_k = x_L\). First note that in any voting equilibrium, \(\alpha^\ell(\tilde{\mathcal{C}}) = (0, 0, 0)\) for all \(\ell \in \mathcal{M}\) since \(G^\ell(\tilde{\mathcal{C}}) = L^\ell(\tilde{\mathcal{C}})\). Also, \(\alpha^L_L(\tilde{\mathcal{C}}) = 1\) and \(\alpha^L_R(\tilde{\mathcal{C}}) = \alpha^L_L(\tilde{\mathcal{C}}) = 0\) for all \(\ell \in \mathcal{N}_R\) since \(G^\ell(\tilde{\mathcal{C}}) = \{R\}\) and \(L^\ell(\tilde{\mathcal{C}}) = \{k, L\}\). Similarly, \(\alpha^h_h(\tilde{\mathcal{C}}) \in \{s, 1\}\) (for \(h = k, L\)) and \(\alpha^h_R(\tilde{\mathcal{C}}) = 0\)
for all $\ell \in \mathcal{N}_L$. This means that in any voting equilibrium - and there exists at least one - $\pi_R(\mathcal{C}, \alpha) = \frac{N_h - M}{N_h - M} \cdot \frac{k}{k}$, while $\pi_h(\mathcal{C}, \alpha) = \frac{N_h - M}{N_h - M} \cdot \frac{k}{k}$ for $h = k, L$, with the inequality strict for at least one of those two candidates. Hence either $W(\mathcal{C}, \alpha) = \{R\}$, or $W(\mathcal{C}, \alpha) = \{h, R\}$ for some $h \in \{k, L\}$, and $U^k(\mathcal{C}, \alpha) < U^k(\mathcal{C}, \alpha)$.

It remains to consider the case where $x_k \in (x_L, x_R)$. Let’s first prove the necessity part. Assume by way of contradiction that there exists an equilibrium profile of voting strategies $\alpha(\mathcal{C})$ such that $U^k(\mathcal{C}, \alpha) < U^k(\mathcal{C}, \alpha)$, but that neither of the two conditions hold. For any equilibrium profile of voting strategies we have $\alpha^L(\mathcal{C}) = \alpha^R(\mathcal{C}) = 0$ and $\alpha^k(\mathcal{C}) = 1$ for all $\ell \in M$ since $G^k(\mathcal{C}) = \{k\}$ and $L^k(\mathcal{C}) = \{L, R\}$. Similarly, $\alpha^L(\mathcal{C}) = 0$ for all $\ell \in N_R$ and $\alpha^R(\mathcal{C}) = 0$ for all $\ell \in N_L$. And by weak undomination, $\alpha^k(\mathcal{C}) \in \{s, 1\}$ for all citizen $\ell$ for whom $k \in G^k(\mathcal{C})$. Hence, in any voting equilibrium $\pi_k(\mathcal{C}, \alpha) \geq [M + (\gamma^L + \gamma^R)] \frac{k}{k}$ and $\pi_h(\mathcal{C}, \alpha) \leq \frac{N_h - M}{N_h - M}$, for $h = L, R$. Now, since condition (2) does not hold, in any voting equilibrium - and there exists one - $W(\mathcal{C}, \alpha) = \{k\}$. And since condition (1) does not hold, $U^k(\mathcal{C}, \alpha) \geq U^k(\mathcal{C}, \alpha)$, a contradiction.

Let’s now prove the sufficiency part. Suppose condition (1) holds. Then, $U^k(\mathcal{C}, \alpha) < U^k(\mathcal{C}, \alpha)$ even if $W(\mathcal{C}, \alpha) = \{k\}$. Suppose condition (2) holds, and construct $\alpha(\mathcal{C})$ as follows: (i) $\alpha^L(\mathcal{C}) = 1$ for all $\ell \in N_h$, for $h = L, R$; (ii) $\alpha^k(\mathcal{C}) = 1$ for all $\ell \in M$; (iii) $\alpha^k(\mathcal{C}) = s$ for all $\ell \in (\mathcal{N}\setminus M)$ for whom $k \in G^k(\mathcal{C})$; and (iv) $\alpha^j(\mathcal{C}) = 0$ otherwise. Hence $W(\mathcal{C}, \alpha) = W(\mathcal{C}, \alpha)$, and $U^k(\mathcal{C}, \alpha) < U^k(\mathcal{C}, \alpha)$.

We are now ready to prove Proposition 4 for the case where vote truncation is allowed. Let $\mathbf{V}$ and $\overrightarrow{\mathbf{V}}$ be any two OVs with $s \in (\mathbf{V}, 1)$. Take $(\mathcal{C}, \alpha)$ a 2-position serious equilibrium under $\mathbf{V}$ - if none exists, then we are done - and call $L$ and $R$ the two candidates. We are now going to construct $(\overrightarrow{\mathcal{C}}, \overrightarrow{\alpha})$ a strongly equivalent serious equilibrium under $\overrightarrow{\mathbf{V}}$. Let $\mathcal{C} \equiv \mathcal{C}$ and $\overrightarrow{\alpha}(\mathcal{C}) \equiv \alpha(\mathcal{C})$. Given that $\mathcal{C}$ is an equilibrium set of candidates, we have $-\frac{\nu_L}{N_L} \geq \delta$ and $-\frac{\nu_R}{N_R} \geq \delta$ - i.e. neither candidate is better off not entering the race. Now, those two conditions hold as well under $\overrightarrow{\mathbf{V}}$ since they do not depend on $s$, and thus neither of those two candidates would be better off not running for election under $\overrightarrow{\mathbf{V}}$. Take $k \in (\mathcal{N}\setminus \mathcal{C})$, and let $\mathcal{C} \equiv \{k, L, R\}$. By Claim 4.4 we know that if $x_k \notin (x_L, x_R)$ there exists an equilibrium profile of voting strategies such that $U^k(\mathcal{C}, \overrightarrow{\alpha}) < U^k(\mathcal{C}, \overrightarrow{\alpha})$. Suppose instead that $x_k \in (x_L, x_R)$. Since $(\mathcal{C}, \alpha)$ is an equilibrium under $\mathbf{V}$ condition (1) and/or condition (2) in Claim 4.4 must hold. If condition (1) holds, then we are done since this condition does not depend on $s$. If condition (2) holds, then so does it under $\overrightarrow{\mathbf{V}}$ since $s < s$. It follows that for any $k \in (\mathcal{N}\setminus \mathcal{C})$ there exists an equilibrium profile of voting strategies $\overrightarrow{\alpha}(\mathcal{C})$ such that $U^k(\mathcal{C}, \overrightarrow{\alpha}) < U^k(\mathcal{C}, \overrightarrow{\alpha})$. For any other set of candidates, take any equilibrium profile of voting strategies. Hence $(\mathcal{C}, \overrightarrow{\alpha})$ is a serious equilibrium under $\overrightarrow{\mathbf{V}}$, and it is strongly equivalent to $(\mathcal{C}, \alpha)$.

Given that condition (2) in Claim 4.4 holding under $\overrightarrow{\mathbf{V}}$ does not imply that it holds under $\mathbf{V}$ as well, we have proved the subset part of the result. The moderation part follows directly from the fact that it is only for moderates that there may not exist an equilibrium profile of voting strategies that deters them from entering the race.

**Proof of Proposition 5.** The 1-position serious equilibrium set is obviously
strONGLY EQUIVALENT WHETHER TRUNCATION IS ALLOWED OR NOT. Moreover, we know from Lemma 3 that 1-position serious equilibria are moderate compared to 2-position serious equilibria. Hence, we only need compare the 2-position serious equilibrium sets. W.l.o.g. let $s > 0$.

Suppose $V$ is worst-punishing. Then the result is trivial since the 2-position serious equilibrium set is empty if voters are not allowed to submit incompletely-ranked ballots.

Suppose $V$ is best-rewarding, and take $(C, \alpha)$ a 2-position serious equilibrium when vote truncation is allowed - if no such equilibrium exists, then we are done. We are going to show that there exists a strongly equivalent equilibrium when vote truncation is not allowed. Let $C = \{L, R\}$. Since $(C, \alpha)$ is an equilibrium it must be that $-\frac{v^k_L}{2} \geq \delta$ and $-\frac{v^k_R}{2} \geq \delta$; that is, neither candidate would be better off not running for election. It remains to show that for any $k \in (N \setminus C)$ there exists an equilibrium profile of voting strategies $\alpha(C \cup \{k\})$ such that $U^k(C \cup \{k\}, \alpha) < U^k(C, \alpha)$. Pick $k \in (N \setminus C)$ arbitrarily. If $x_k \notin (x_L, x_R)$, then we know from Claims 4.1 and 4.2 that such a profile of voting strategies exists. If $x_k \in (x_L, x_R)$, we know from Claim 4.4 and $(C, \alpha)$ a serious equilibrium that either $-\left(-\frac{v^k_L + v^k_R}{2}\right) < \delta$, or $(\gamma^k_L + \gamma^k_R) s \leq \frac{N - 3M - 2}{2}$ (or both). If the former, then we are done. Instead, suppose that only the latter holds, and assume w.l.o.g. that $x_k \in (x_L, m]$. First note that condition (iii) of Claim 4.3 holds. To see this, assume the contrary. Then,

$$
\left(2\gamma^k_L + \gamma^k_R - \frac{N + M - 2}{2}\right) s \geq \frac{N - 3M - 2}{2} \geq (\gamma^k_L + \gamma^k_R) s
$$

and thus $[(\gamma^k_L - \frac{N-M}{2}) + (1 - M)] \geq 0$, a contradiction. Moreover, either condition (b) of Claim 4.3 holds, or condition (c) (or both). To see this, suppose the contrary. First note that the second inequality in (c) necessarily holds since otherwise we would have

$$
\left(\gamma^k_L + 2\gamma^k_R - \frac{N - M + 2}{2}\right) s \geq \frac{N - 3M - 2}{2} \geq (\gamma^k_L + \gamma^k_R) s
$$

and thus $[(\gamma^k_R - \frac{N-M}{2}) - 1] \geq 0$, a contradiction.

It must then be that $\frac{M_s}{2} < (M + \gamma^k_R - \gamma^k_L)$, or $(M + \gamma^k_R + 1) s > (1 + \gamma^k_s)$. Together with condition (b) not holding, we have

$$
\left(\gamma^k_L + 2\gamma^k_R - \frac{N - M}{2}\right) s > \frac{N - 3M - 2}{2} \geq (\gamma^k_L + \gamma^k_R) s
$$

and thus $(\gamma^k_R - \frac{N-M}{2}) > 0$, a contradiction.

Hence, when vote truncation is not allowed there exists a serious equilibrium which is strongly equivalent to $(C, \alpha)$. Now, given that $(C, \alpha)$ was taken arbitrarily this holds true for any such equilibrium. Thus the subset part of the result. And the subset is moderate since the only difference lies in whether the entry by moderates can be deterred.

**Proof of Proposition 6.** Recall that when vote truncation is allowed, serious contenders are located on at most 2 platforms. Moreover, the 1-position serious equilibrium sets are strongly equivalent and are moderate compared to the 2-position serious equilibrium sets (by Lemma 3). Hence, we only need to show that the
Let \( \mathcal{C} = \{c_1, \ldots, c_{N-1}\} \) denote the vector of scores a voter can cast. If all candidates are located on a single platform, then the result is trivial.

Suppose candidates are located on at least three platforms. Take \( k \in \mathcal{C} \) such that there exists \( x_k \in (x_L, x_R) \) for whom \( \gamma_L^k + \gamma_R^k \geq N-3M-2 \) s that there may not exist a serious equilibrium under \( V \) which is strongly equivalent to a 2-position serious equilibrium under \( \tilde{V} \). Hence the moderate result since \( \gamma_L^k + \gamma_R^k \) increases with the distance between \( x_L \) and \( x_R \).

**Proof of Lemma 2.** Take \( \mathcal{C} \) a non-empty set of candidates, and let \( s = (s_1, s_2, \ldots, s_N) \) denote the vector of scores a voter can cast. If all candidates are located on a single platform, then the result is trivial.

Suppose candidates are located on at least three platforms. Take \( k \in \mathcal{C} \) such that there exist \( i \) and \( j \) in \( \mathcal{C} \) with \( x_i < x_k < x_j \), and define \( \mathcal{C}_k = \{c \in \mathcal{C} : x_k = x_k\} \).

Define \( \lambda : \mathcal{C} \rightarrow \{1, \ldots, \#\mathcal{C}_k\} \) (for \( h = L, R \)) one-to-one and onto. Construct \( \alpha(\mathcal{C}) \) as follows: (1) for all \( \ell \in \mathcal{N}_h \), let \( \alpha^h(\mathcal{C}) = s_{\lambda_h(\ell)} \) for all \( \ell \in \mathcal{C}_h \); and (2) let \( \alpha^h(\mathcal{C}) = 0 \) otherwise. Hence, \( \alpha(\mathcal{C}) \) is an equilibrium profile of relatively undominated voting strategies.

Suppose candidates are located on at least three platforms. Take \( k \in \mathcal{C} \) such that there exist \( i \) and \( j \) in \( \mathcal{C} \) with \( x_i < x_k < x_j \), and define \( \mathcal{C}_k = \{c \in \mathcal{C} : x_k = x_k\} \).

Define \( \lambda : \mathcal{C} \rightarrow \{1, \ldots, \#\mathcal{C}_k\} \) one-to-one and onto. For all citizen \( \ell \), let \( Y^\ell \equiv \{i \in (\mathcal{C} \setminus \mathcal{C}_k) : v_i^\ell \geq v_k^\ell\} \), and for a non-empty \( Y^\ell \) define \( \lambda^\ell : Y^\ell \rightarrow \{\#\mathcal{C}_k + 1, \ldots, \#(\mathcal{C}_k \cup Y^\ell)\} \) one-to-one and onto. Note that \( Y^\ell = \emptyset \) for all citizen \( \ell \) with \( x_\ell = x_k \). Also, for any citizen \( \ell \) with \( x_\ell < x_k \) we have \( x_i < x_i (x_i > x_i, \text{resp.}) \) for all \( i \in Y^\ell \). Construct \( \alpha(\mathcal{C}) \) as follows: (1) for all citizen \( \ell \), let \( \alpha^\ell(\mathcal{C}) = s_{\lambda^\ell(i)} \) for all \( i \in \mathcal{C}_k \) and \( s_{\lambda^\ell(i)} \) for all \( i \in Y^\ell \); and (2) let \( \alpha^\ell(\mathcal{C}) = 0 \) otherwise. Hence, \( W(\mathcal{C}, \alpha) \subseteq \mathcal{C}_k \) and nobody is pivotal for a candidate \( i \) with \( x_i \neq x_k \) since he is at least 2 votes short from the winning candidate(s) - recall that no citizen at or to the right of \( x_k \) (left of \( x_k \), resp.) casts a vote for a candidate \( i \) with \( x_i < x_k \) (\( x_i > x_k \), resp.).
Proof of Proposition 7. (I) Take \( V \) a MV with \( q > q^* \), where \( q^* \) is defined as follows. For any \( x_L \) and \( x_R \) in \( X \), \( x_L \neq x_R \), with \( v^n_L = v^n_R \) and \( n_h \neq 0 \) (for \( h = L, R \)) define \( c_h \in \mathbb{N} \) (for \( h = L, R \)) such that

\[
\begin{align*}
\text{given } c_R, \ c_L \text{ solves } & -\frac{c_R}{c(c-1)}v^L_R \geq \delta > -\frac{c_R}{c(c+1)}v^L_R, \text{ and} \\
\text{given } c_L, \ c_R \text{ solves } & -\frac{c_L}{c(c-1)}v^R_L \geq \delta > -\frac{c_L}{c(c+1)}v^R_L 
\end{align*}
\]

, where \( c \equiv (c_L + c_R)^52 \). Define \( q(x_L, x_R) \equiv \max \{c_L, c_R\} \), and let \( q^* \) be the largest \( q(x_L, x_R) \).

Recall that in any election where voters are allowed to truncate their ballot serious contenders are located on at most two platforms.

Take \((C, \alpha)\) a 1-position serious equilibrium. We know that \( C = \{i\} \). If \( x_i = m \), then the result holds trivially. Suppose instead that \( x_i \neq m \). Given \((C, \alpha)\) an equilibrium, there does not exist a citizen \( h \) such that \( v^m_h > v^m_i \) and \( -v^i \geq \delta \). Now, this applies to \( h = m \). And since \( v^m_m = 0 > v^m \), it must then be that \( -v^m < \delta \).

Take \((C, \alpha)\) a 2-position serious equilibrium, and let \( x_L \) and \( x_R \) - with \( x_L < x_R \) - denote the two platforms on which the candidates are running. Pick a citizen \( k \) with \( x_k = m \), and let \( \mathcal{C} \equiv \mathcal{C} \cup \{k\} \). Then in any voting equilibrium, \( W(\mathcal{C}, \alpha) = \{k\} \).

To see why, assume by way of contradiction that there exists \( i \in W(\mathcal{C}, \alpha), i \neq k \). First, note that for all citizen \( \ell \) with \( x_\ell \geq m \), \( L'(\mathcal{C}) = C_L \), and thus \( \alpha'_L(\mathcal{C}) = 0 \) for all \( i \in C_L \) (by weak undominance). Hence, in any voting equilibrium \( \pi_i(\mathcal{C}, \alpha) \leq \frac{N+2}{2} \). Similarly, for all \( j \in C_R \). Also, by definition of \( q^* \) we have that \( c_h \leq q^* \), for \( h = L, R \). Let \( \sigma \equiv \frac{n_L x_L + n_R x_R}{n_L + n_R} \), where \( \eta_h \equiv \# \{i \in W(\mathcal{C}, \alpha) : x_i = x_h\} \), for \( h = L, k, R \).

By strict concavity of \( u^\ell() \), we have \( u^\ell(\sigma) \geq V^\ell(\mathcal{C}, \alpha) \) for all citizen \( \ell \), with a strict inequality whenever there exist two candidates \( i \) and \( j \) in \( W(\mathcal{C}, \alpha) \), with \( x_i \neq x_j \). Either \( m \leq \sigma \), in which case \( v^\ell_R > V^\ell(\mathcal{C}, \alpha) \) for all citizen \( \ell \) with \( x_\ell \leq m \), and \( \alpha'_{L}(\mathcal{C}) = 1 \) (by Relative Undominance and \( q > q^* \geq c_L \)). Or \( m > \sigma \), in which case \( v^\ell_L > V^\ell(\mathcal{C}, \alpha) \) for all citizen \( \ell \) with \( x_\ell \geq m \), and \( \alpha'_{L}(\mathcal{C}) = 1 \) (by Relative Undominance and \( q > q^* \geq c_R \)). In both cases, \( \pi_k(\mathcal{C}, \alpha) \geq \frac{N+2}{2} \). But then, \( \pi_k(\mathcal{C}, \alpha) > \pi_i(\mathcal{C}, \alpha) \) for all candidate \( i \), \( i \neq k \), which contradicts our assumption that there exists \( i \in W(\mathcal{C}, \alpha), i \neq k \). Hence \( W(\mathcal{C}, \alpha) = \{k\} \), and for \((C, \alpha)\) to be an equilibrium it must then be that \( -v^m < \delta \) for all candidate \( i \).

(II) Take \( V \) an OV. Let \( s \equiv \frac{1}{2} \left( \frac{N-3M-2}{N-M-1} \right) \) and \( \tilde{s} \equiv \left( \frac{N-3M-2}{N-M-2} \right) \). We are going to prove the result via the following three claims.

Claim 7.1. Let \( V \) and \( \tilde{V} \) be any two OV, and suppose that citizens are allowed to cast incompletely-ranked ballots. Let \( s \) and \( \tilde{s} \) denote the second-place scores in any 3-candidate race under \( V \) and \( \tilde{V} \), respectively. If \( s > \tilde{s} \), then the relatively undominated serious equilibrium set under \( V \) is a subset of the relatively undominated serious equilibrium set under \( \tilde{V} \).

Proof of Claim 7.1. Take \((C, \alpha)\) a relatively undominated serious equilibrium under \( V \). We are going to construct \((\mathcal{C}, \alpha)\) a strongly equivalent relatively undominated serious equilibrium under \( \tilde{V} \).

\(^{52}\)If there is no \( \{c_L, c_R\} \in \mathbb{N}^2 \) that solves \((*)\), then let \( c_L = c_R = 0 \). Also, if for any \( x_L \) and \( x_R \) in \( X \) either \( v^L_L \neq v^R_R \), or \( n_h = 0 \) for some \( h \in \{L, R\} \), then let \( q^* = 0 \).
Either $\mathcal{C} = \{i\}$, in which case it is straightforward. Or $\mathcal{C} = \{L, R\}$ with $x_L < x_R$. Let $\hat{\alpha}(\mathcal{C}) = \alpha(\mathcal{C})$ and $\hat{\alpha}(\mathcal{C}, \{h\}) = \alpha(\mathcal{C} \setminus \{h\})$, for $h = L, R$. Pick $k \in (N \setminus \mathcal{C})$, and let $\tilde{C} \equiv (\mathcal{C} \cup \{k\})$. If $x_k \leq x_L$, then construct $\hat{\alpha}(\tilde{C})$ as follows: (1) for all $\ell \in N_h$, let $\hat{\alpha}_h^\ell(\tilde{C}) = 1$ (for $h = L, R$); (2) for all citizen $\ell$ with $v_h^\ell > v_R^\ell$, let $\hat{\alpha}_R^\ell(\tilde{C}) = \hat{\ell}$; (3) for all $\ell \in M$, let $\hat{\alpha}_R^\ell(\tilde{C}) = \hat{\ell}$ and $\hat{\alpha}_R^\ell(\tilde{C}) = 1$ if $x_k < x_L$; and (4) let $\hat{\alpha}_i^\ell(\tilde{C}) = 0$ otherwise. Hence $\hat{\alpha}(\tilde{C})$ is an equilibrium profile of relatively undominated voting strategies, and $U^k(\tilde{C}, \alpha) < U^k(\mathcal{C}, \hat{\alpha})$. Similarly if $x_k \geq x_R$. Suppose $x_k \in (x_L, m]$. Then there exists $\hat{\alpha}(\tilde{C})$ an equilibrium profile of relatively undominated voting strategies such that $W(\tilde{C}, \hat{\alpha}) = W(\mathcal{C}, \alpha)$. To see this, consider the following (exhaustive) four cases:

**Case 1:** $W(\tilde{C}, \hat{\alpha}) = \{R\}$. By Relative Undominance it must be that $\pi_R(\tilde{C}, \alpha) \leq \frac{N-M}{2}$ since $R \in L^\ell(\tilde{C})$ for all citizen $\ell$ with $x_\ell \leq m$, $\pi_L(\tilde{C}, \alpha) = (\mu + \nu s)$ for some $\mu, \nu \in \mathbb{Z}_+$ with $(\mu + \nu) = \frac{N-M}{2}$, and $\pi_k(\tilde{C}, \alpha) \geq [M + (\nu + \mu s) + \gamma_k]$ where $\gamma_k$ is defined as in the proof of Proposition 4. Construct $\hat{\alpha}(\tilde{C})$ as follows: (1) for all citizen $\ell$ with $x_\ell \leq m$, let $\hat{\alpha}_L^\ell(\tilde{C}) = \alpha_L^\ell(\tilde{C})$; (2) for all $\ell \in N_R$, let $\hat{\alpha}_R^\ell(\tilde{C}) = 1$; and, if $v_R^\ell \geq v_R^\ell$, let $\hat{\alpha}_R^\ell(\tilde{C}) = \hat{\ell}$ (for $h = L, R$); and (3) let $\hat{\alpha}_i^\ell(\tilde{C}) = 0$ otherwise. It follows that $\pi_L(\tilde{C}, \hat{\alpha}) < \pi_L(\tilde{C}, \alpha)$, $\pi_k(\tilde{C}, \hat{\alpha}) < \pi_k(\tilde{C}, \alpha)$, and $\pi_R(\tilde{C}, \hat{\alpha}) > \pi_R(\tilde{C}, \alpha)$. Hence $W(\tilde{C}, \hat{\alpha}) = \{R\}$, and $\tilde{\alpha}(\tilde{C})$ is an equilibrium profile of relatively undominated voting strategies.

**Case 2:** $W(\tilde{C}, \hat{\alpha}) = \{L\}$. Similar to case 1.

**Case 3:** $W(\tilde{C}, \hat{\alpha}) = \{k\}$. Construct $\hat{\alpha}(\tilde{C})$ as follows: (1) for all citizen $\ell$, let $\hat{\alpha}_L^\ell(\tilde{C}) = 1$; (2) for all citizen $\ell$ with $v_R^\ell \geq v_R^\ell$, let $\hat{\alpha}_R^\ell(\tilde{C}) = \hat{\ell}$ (for $h = L, R$); and (3) let $\hat{\alpha}_i^\ell(\tilde{C}) = 0$ otherwise. Hence $W(\tilde{C}, \hat{\alpha}) = \{k\}$, and $\tilde{\alpha}(\tilde{C})$ is an equilibrium profile of relatively undominated voting strategies.

**Case 4:** $W(\tilde{C}, \hat{\alpha}) = \{L, R\}$. It is easy to see that since $\alpha(\tilde{C})$ is an equilibrium profile of relatively undominated voting strategies it must be that $\pi_h(\tilde{C}, \alpha) = \frac{N-M}{2}$ (for $h = L, R$), and $\pi_k(\tilde{C}, \alpha) = (M + \mu s)$ with $\mu \in \mathbb{Z}_+$ such that $\frac{N-M}{2} \leq \mu \leq (N-M)$. Let $\hat{\alpha}(\tilde{C}) = \alpha(\tilde{C})$. Hence $\pi_h(\tilde{C}, \alpha) = \pi_h(\tilde{C}, \alpha)$ (for $h = L, R$), and $\pi_k(\tilde{C}, \alpha) = \pi_k(\tilde{C}, \alpha)$. It then follows that $W(\tilde{C}, \hat{\alpha}) = W(\tilde{C}, \alpha)$.

Similarly if $x_k \in (m, x_R]$. Finally, for any other set of candidates $\mathcal{C}$ let $\hat{\alpha}(\mathcal{C})$ be any equilibrium profile of relatively undominated voting strategies (which we know to exist by Lemma 2). Hence $(\mathcal{C}, \hat{\alpha})$ is a relatively undominated serious equilibrium under $\tilde{V}$, and is strongly equivalent to $(\mathcal{C}, \alpha)$. ||
Claim 7.2. Suppose the election is held under an OV, and citizens are allowed to cast incompletely-ranked ballots. Let \( s \) denote the second-place score in a 3-way race. If \( s < \bar{s} \), then the relatively undominated serious equilibrium set is equivalent to the serious equilibrium set.

Proof of Claim 7.2. Take \( V \) an OV with \( s < \bar{s} \). We know that in any serious equilibrium either \( \mathcal{C} = \{1\} \), or \( \mathcal{C} = \{L, R\} \) with \( x_L < x_R \) and \( v^m_L = v^m_R \). Now since any relatively undominated serious equilibrium is a serious equilibrium we only need to show that for any serious equilibrium there exists a strongly equivalent relatively undominated serious equilibrium.

Take \((\mathcal{C}, \alpha)\) a serious equilibrium. If \#\( \mathcal{C} = 1 \), then the result is obvious. Suppose instead that \( \mathcal{C} = \{L, R\} \) with \( x_L < x_R \). First note that for any given set of candidates \( \mathcal{C} \) with \#\( \mathcal{C} \in \{1, 2\} \), \( \alpha(\mathcal{C}) \) is weakly undominated if and only if it is relatively undominated. Hence \( \alpha(\mathcal{C}) \) and \( \alpha(\mathcal{C}\setminus\{h\}) \) (for \( h = L, R \)), are equilibrium profiles of relatively undominated voting strategies. Now pick \( k \in (\mathcal{N}\setminus\mathcal{C}) \), and let \( \mathcal{C}' \equiv (\mathcal{C} \cup \{k\}) \). If \( x_k \in [x_L, x_R] \), then construct \( \alpha(\mathcal{C}') \) as follows: (1) for all \( \ell \in \mathcal{N}_h \) (for \( h = L, R \)), let \( \alpha'_k(\mathcal{C}') = 1 \) and \( \alpha'_k(\mathcal{C}') = s \) if \( v'_k \geq V^\ell(\mathcal{C}, \alpha) \); (2) for all \( \ell \in \mathcal{M} \), let \( \alpha'_k(\mathcal{C}') = 1 \) if \( x_k \in (x_L, x_R) \); and (3) let \( \alpha'_k(\mathcal{C}') = 0 \) otherwise. Hence \( \pi_L(\mathcal{C}', \alpha) = \pi_R(\mathcal{C}', \alpha) = \frac{N-M}{2} \) and \( \pi_k(\mathcal{C}', \alpha) \leq [M + (N - M) s] \). Now, since \( s < \bar{s} \) we have \( \pi_h(\mathcal{C}', \alpha) > \frac{\pi_k(\mathcal{C}', \alpha) + (1-s)}{2} \) (for \( h = L, R \)), and thus \( \alpha(\mathcal{C}') \) is an equilibrium profile of relatively undominated voting strategies with \( W(\mathcal{C}, \alpha) = W(\mathcal{C}, \alpha) \). Instead, if \( x_k \notin [x_L, x_R] \) - w.l.o.g. suppose \( x_k < x_L \) - construct \( \alpha(\mathcal{C}') \) as follows: (1) for all \( \ell \in \mathcal{N}_h \) (for \( h = L, R \)), let \( \alpha'_k(\mathcal{C}') = 1 \) and \( \alpha'_k(\mathcal{C}') = s \) if \( v'_k \geq v'_L \); (2) for all \( \ell \in \mathcal{M} \), let \( \alpha'_k(\mathcal{C}') = s \) and \( \alpha'_R(\mathcal{C}') = 1 \); (3) let \( \alpha'_k(\mathcal{C}') = 0 \) otherwise.

Again \( \alpha(\mathcal{C}') \) is an equilibrium profile of relatively undominated voting strategies with \( W(\mathcal{C}', \alpha) = \{R\} \). In any case, \( U^k(\mathcal{C}', \alpha) < U^k(\mathcal{C}, \alpha) \). Finally, for any other set of candidates \( \mathcal{C} \) let \( \alpha(\mathcal{C}') \) be any equilibrium profile of relatively undominated voting strategies - which we know to exist by Lemma 2. Hence \( (\mathcal{C}, \alpha) \) is a relatively undominated serious equilibrium. ||

Claim 7.3. Suppose the election is held under an OV, and citizens are allowed to cast incompletely-ranked ballots. Let \( s \) denote the second-place score in a 3-candidate race. If \( s > \bar{s} \), then in any relatively undominated serious equilibrium we have \(-v^m_i < \delta \) for every candidate \( i \).

Proof of Claim 7.3. Similar to the proof in part (I). ||