Multi-Product Serial Cost Sharing: 
An Incompatibility Result

Cyril Téjédo       Michel Truchon
CRÉFA and Département d’économique
Université Laval

cahier 9919

cahier 99-13
du Centre de Recherche en Économie et Finance Appliquées
CRÉFA

November 1999
Abstract

The Serial Cost Sharing Rule has been conceived originally for problems where agents ask for different quantities of an homogeneous private good, the sum of which is produced by a single facility. Two important features of this rule is the equal treatment of equal demands and the protection it offers to smaller demanders against larger ones. A natural question is whether the Serial Cost Sharing Rule can be extended to a problem where agents demand more than one good, while keeping its interesting properties. We show that the equal treatment of equal demands and some form of protection of small demanders against larger ones, called the serial principle, are incompatible in a more general context.

Keywords: Cost Sharing, Serial Principle, Equal Treatment of Equals

JEL Classification: D63, C71

Résumé

La règle de partage séquentiel des coûts a été conçue, à l’origine, pour le cas où les demandes des agents portent sur un bien privé homogène, produit par une technologie non reproductible. Les deux caractéristiques essentielles de cette règle sont le traitement égalitaire des égaux et la protection des petits demandeurs contre les plus grands. Une question naturelle est de savoir si la règle de partage séquentiel peut être généralisée au cas où les agents demandent plus d’un bien, tout en conservant ces deux caractéristiques. Nous montrons que le traitement égalitaire des égaux et une forme de protection des petits demandeurs contre les plus grands, appelée le principe séquentiel, sont incompatibles dans ce contexte plus général.
1 Introduction

The Serial Cost Sharing Rule has received much attention since its introduction by Shenker (1990) and its extensive analysis by Moulin and Shenker (1992, 1994). This rule has been conceived originally for problems where agents ask for different quantities of an homogeneous private good, the sum of which is produced by a single facility. Its two characteristic features are the equal treatment of equal demands (ETE) and the protection it offers to smaller demanders against larger ones (ILD).

(ILD) and (ETE) are not much compelling in a more general context since demands may not be comparable. Sprumont (1998) proposes the Serial Principle (SP) and a Symmetry condition (S) as natural extensions of respectively (ILD) and (ETE). (SP) says that an agent who pays less than another agent should not see her cost share change if this other agent increases his demand. Sprumont also exhibits an extension of the Serial Cost Sharing Rule for the context where each agent demands a single good but where the goods are heterogeneous across agents. His Axial Serial Rule, as it is called, satisfies (SP) and (S).

Koster, Tijs, and Borm (1998) proposes the same kind of extension, which they call the Radial Serial Rule, for the context where agents demand baskets of many homogeneous and private goods. They show that (SP) and (ETE), which is equivalent to (S) in this context, are not compatible with three others conditions (independence of null agents, rank independence with respect to irrelevant agents and some form of scale independence).

We reinforce their result by showing that the incompatibility is fundamentally between (SP) and (ETE). We also argue that, while (ETE) is weak in such a general context, (SP) alone has a weak ethical content. Our result thus shows that two very weak properties are incompatible in the context considered.
2 The Cost Sharing Problem

Throughout this note, $N = \{1, \ldots, n\}$ is a fix set of agents and $M = \{1, \ldots, m\}$ a fixed and finite set of private and divisible commodities. The agents jointly own a facility to jointly produce any vector of commodities that is demanded.

A particular level of output is described by a vector $y \in \mathbb{R}_+^m$. The demand of agent $i$ is described by a vector $q_i \in \mathbb{R}_+^m$. The scalar $q_{ih}$ is the demand of commodity $h$ by agent $i$. A profile of demands is an element $Q \in \mathbb{R}_+^{nm}$, which can be written as a matrix:

$$Q = \begin{bmatrix}
q_{11} & \cdots & q_{1m} \\
\vdots & & \vdots \\
q_{n1} & \cdots & q_{nm}
\end{bmatrix}$$

Since commodities are private, aggregate demand is given be the summation of individual demands. Let $e = (1, \ldots, 1) \in \mathbb{R}^n$. The aggregate demand is the product $eQ$ where $e$ must be read as a row-vector.

Let $\mathbb{C}(m)$ be the set of continuous and non-decreasing functions $c : \mathbb{R}_+^m \to \mathbb{R}_+$ satisfying $c(0) = 0$. A cost sharing problem is a list $(Q, c) \in \mathbb{R}_+^{nm} \times \mathbb{C}(m)$. A cost sharing rule is a function $\xi : \mathbb{R}_+^{nm} \times \mathbb{C}(m) \to \mathbb{R}_+^n$ satisfying the budget balance condition $\sum_{i \in N} \xi_i(Q, c) = c(eQ)$. The vector $\xi(Q, c)$ is the list of cost shares for the problem $(Q, c)$.

**Definition 1** A cost sharing rule $\xi : \mathbb{R}_+^{nm} \times \mathbb{C}(m) \to \mathbb{R}_+^n$ satisfies Equal Treatment of Equals (ETE) if for all $(Q, c) \in \mathbb{R}_+^{nm} \times \mathbb{C}(m)$ and $i, j \in N$, the following holds,

$$q_i = q_j \Rightarrow \xi_i(Q, c) = \xi_j(Q, c)$$

**Definition 2** A cost sharing rule $\xi : \mathbb{R}_+^{nm} \times \mathbb{C}(m) \to \mathbb{R}_+^n$ satisfies Independence of Larger Demands (ILD) if for two cost sharing problems $(Q, c)$ and $(Q', c) \in \mathbb{R}_+^{nm} \times \mathbb{C}(1)$ and any $i \in N$ such that such that $q'_i = q_i$ and

$$q'_j = q_j \quad \forall j \in N \setminus \{i\} : q_j < q_i$$

$$q'_j \geq q_j \quad \forall j \in N \setminus \{i\} : q_i \leq q_j$$

3
the following holds:

\[ \xi_i (Q, c) = \xi_i (Q', c) \]

In the context considered here, (ILD) is not much compelling since the relation \( \leq \) on \( \mathbb{R}^m \) is not complete. Not all demands can be compared. To obviate this problem, Sprumont (1998) proposes that the demands be ordered according to the cost shares produced by the cost sharing rule itself. This yields the Serial Principle, which requires that an agent’s cost share be unaffected by increases in the demands of those who initially pay more than him.

**Definition 3** A cost sharing rule \( \xi : \mathbb{R}^{nm}_+ \times \mathcal{C}(m) \to \mathbb{R}^n_+ \) satisfies the Serial Principle (SP) if for two cost sharing problems \((Q, c)\) and \((Q', c)\) \(\in \mathbb{R}^{nm}_+ \times \mathcal{C}(m)\) and any \(i \in N\) such that \(q_i' = q_i\) and

\[
q_j' = q_j \quad \forall j \in N \setminus \{i\} : \xi_j (Q, c) < \xi_i (Q, c) \\
q_j' \geq q_j \quad \forall j \in N \setminus \{i\} : \xi_i (Q, c) \leq \xi_j (Q, c)
\]

the following holds:

\[ \xi_i (Q, c) = \xi_i (Q', c) \]

### 3 The Incompatibility Result

We are now ready to prove our result.

**Theorem 1** For \( m \geq 2 \), there does not exist a cost sharing rule \( \xi : \mathbb{R}^{nm}_+ \times \mathcal{C}(m) \to \mathbb{R}^n_+ \) that satisfies Equal Treatment of Equals (ETE) and the Serial Principle (SP).

**Proof.** Let \( \xi \) satisfies (ETE) and (SP) and consider two profiles of demands \( Q, \tilde{Q} \in \mathbb{R}^{nm}_+ \) and a cost function \( c \in \mathcal{C}(m) \):

\[ q_1 = \tilde{q}_1 \leq \tilde{q}_j, \quad j = 2, \ldots, n \quad (1) \]
\[ q_j \leq \tilde{q}_j, \; j = 2, \ldots, n \]  

(2)

\[ \xi_i (Q, c) \leq \xi_j (Q, c) \quad \forall i, j \in N : i < j. \]  

(3)

Note that (3) implies:

\[ \xi_1 (Q, c) \leq \frac{c(eQ)}{n} \]  

(4)

Next, consider the following profile of demands \( Q^1 = (q_1, \ldots, q_1) \). By (ETE), we get:

\[ \xi_1 (Q^1, c) = \frac{c(nq_1)}{n} \]  

(5)

By (SP), (1) and (3) imply:

\[ \xi_1 (\tilde{Q}, c) = \xi_1 (Q^1, c) \]  

(6)

Similarly, (2) and (3) imply:

\[ \xi_1 (\tilde{Q}, c) = \xi_1 (Q, c) \]  

(7)

Combining (4)-(7), we get:

\[ c(eQ) \geq c(nq_1) \]  

(8)

This last inequality says that aggregate demand \( eQ \) should belong to the upper contour set of \( c \) generated by \( nq_1 \). This is not the case for all cost functions. Consider for example the two profiles:

\[ Q = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad \tilde{Q} = \begin{bmatrix} 2 & 1 \\ 2 & 2 \end{bmatrix} \]

We have \( eQ = (3, 3) \) and \( 2q_1 = (4, 2) \). Inequality (8) is violated with the two classes of cost functions defined respectively by

\[ c(x_1, x_2) = \tau (x_1^3 + x_2^3) \]

and

\[ c(x_1, x_2) = \tau (2x_1 + x_2) \]

where \( \tau : \mathbb{R}_+ \to \mathbb{R}_+, \tau' (\cdot) > 0 \), and otherwise arbitrary. \[ \blacksquare \]
Remark 2 There exist rules that satisfy (ETE) without (SP), for example the Radial Serial Rule of KTB and the Shapley-Shubik rule. There are also rules that satisfy (SP) without (ETE). For example, suppose that agents are ordered in some arbitrary way. Then consider the formula:

$$\xi_{i}^{SP}(Q, c) = \begin{cases} 
  c(eQ) & \text{if } q_1 = q_2 = \ldots = q_{i-1} = 0 \text{ and } q_i > 0 \\
  0 & \text{otherwise}
\end{cases}$$

$\xi^{SP}$ satisfies (SP) but not (ETE). Interestingly, $\xi^{SP}$ is not a Serial Extension.

4 Conclusion

We showed that, in a multi-product context, Equal Treatment of Equals and the Serial Principle are incompatible. Thus only one of the two properties can be imposed on a rule. We also showed that the Serial Principle alone has a weak ethical content. As mentioned in the introduction, Equal Treatment of Equals is also very weak in the same context. Thus our theorem shows that two very weak properties are incompatible in this context.

References


