A Preference Regime Model of Bull and Bear Markets*

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Abstract

This paper develops a consumption-based asset pricing model in which attitudes towards risk are contingent upon the state of the world. For a low (high) level of consumption relative to a subjective metric, counter-cyclical (pro-cyclical) risk aversion implies that consumption shocks generate larger fluctuations in marginal utility, against which the agent will hedge in choosing his optimal portfolio. Asset prices are studied using two-state Markov preference regimes where bull and bear markets reflect alternating periods of low and high risk aversion. Joint estimation of bond and stock prices highlights moderate and infrequent movements in risk aversion, and a marked improvement on the model’s ability to capture the cyclical nature of observed asset prices.

Keywords: Asset pricing models, Bayesian analysis, excess volatility, Markov chain, regime switching, risk aversion, state-dependent preferences.

JEL classification: C110, G120

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1 Introduction

The main proposition of the consumption-based capital asset pricing model (C-CAPM) is that a risk-averse representative agent with state- and time-separable preferences allocates consumption and portfolio choices so as to smooth out risks to the inter-temporal marginal rates of substitution (IMRS) over uncertain consumption streams (Lucas 1978, Breeden 1979). The intuition behind this result is straightforward: assets that are expected to pay high returns when marginal utility is high will tend to be valued more by the agent than those whose returns are negatively correlated with anticipated marginal utility. Hence, under the C-CAPM, stock prices reflect the present value of anticipated dividends, where the stochastic discount factor is the IMRS between consumption in the current period and consumption in the periods where dividends are paid out; bond prices are based on the conditional mean of IMRS. Furthermore, if within-period utility is iso-elastic, the model predicts that expected excess returns for risky assets are equal to their covariance with consumption (the quantity of risk) times the coefficient of relative risk aversion (the price of risk).

The empirical difficulties of the C-CAPM are well documented (Kocherlakota 1996, Campbell 1996, Cochrane 1997, provide enlightening surveys). First, the moderate quantity of consumption risk does not warrant the high average excess returns we observe, unless an unreasonably large price of risk is assumed (the equity premium puzzle). Secondly, a high curvature coefficient for within-period utility reduces the average MRS; because observed bond prices are high, the subjective discount factor must be increased to levels greater than one (the risk-free rate puzzle). Third, stock prices are quite cyclical relative to the discounted dividends stream while the excess returns are distressingly predictable, even allowing for time-varying quantities of risk (the predictability puzzle).

Recent research focuses on time-varying prices of risk as a potential source of mis-specification. In particular, counter-cyclicality in risk aversion is introduced through wealth dependencies (Bakshi and Chen 1996) and through time-varying habits in which the bliss point is a function of past consumption (Campbell and Cochrane 1995). Both models can be seen as special cases of state-dependent preferences in which attitudes are determined by contemporary state variables, whose evolution may or may not be affected by
the agent’s decisions. These preferences have been advocated in other settings for the study of changing, or apparently excessive risk aversion, e.g., Karni (1983, 1987). Unfortunately, their empirical success within the context of the C-CAPM has been mixed. Implementation of the Bakshi and Chen (1996) model requires uncontroversial proxies for aggregate wealth — which are difficult to obtain — and the degree of non-linearities involved forces the use of restricted cases, thus abandoning much of the potential for improvement over the standard model. Moreover, relying on consumption volatility to explain time-varying risk aversion also limits the potential gains: since aggregate consumption remains frustratingly smooth, time-varying habits — albeit useful for addressing the risk-free rate puzzle — do not generate sufficient additional risk to reconcile the high premia with reasonable prices of risk.

We adopt what might be considered an unrestricted approach to time-varying prices of risk in which the state variable that underlies the representative agent’s preferences is treated as a latent variable. In our model, within-period utility is conditionally (upon the prevailing preference state) iso-elastic, and we ask what pattern of attitudes toward risk might be consistent with observed movements of asset prices. When risk aversion is state-dependent, low (high) consumption and counter-cyclical (pro-cyclical) curvature imply much more dramatic movements in marginal utility. These fluctuations affect the agent’s choice over risk-free and risky assets in hedging against IMRS risk.

In our discussion of asset prices, we develop a simple discrete-time model in which risk aversion follows a two-state Markov process. This specification has an intuitive interpretation: the celebrated ‘bear’ and ‘bull’ market classification of asset market activity is produced as attitudes toward risk shift between preference regimes of high and low risk aversion. The results suggest a marked improvement in reproducing the joint first and second moments of stock and bond prices, as well as being quite useful in addressing the cyclical nature of equity prices. These estimated parameters point toward (i) realistic levels of risk aversion, and (ii) infrequent, moderate — though significant — and counter-cyclical shifts in curvature. The main result therefore comes as good news for the standard model: within a given regime, the conventional C-CAPM performs surprisingly well in reproducing asset prices; its inability to reproduce the financial market cycle might stem from the inadequate restriction that risk aversion is constant throughout the sample.
We also show that state-dependent preferences have the potential to successfully resolve the equity premium, the risk-free rate and the predictability puzzles. First, excess returns reflect two independent contributors to IMRS risk: consumption and curvature risk (i.e. the covariance between risk preferences and returns); this second element justifies high risk premia without requiring excessive risk aversion. Furthermore, the loading for the concavity risk is a function of consumption, which has the potential for explaining cyclical movements in returns. Finally, counter-cyclicality of risk aversion, as well as the additional volatility in the IMRS induce a precautionary demand for risk-free assets, which would justify high bond prices without requiring negative discounting of future utility.

We conclude with the familiar calls for further research: relaxing the hypothesis of fixed attitudes toward risk appears to be a step in the right direction, but we need to consider which variables might explain these movements. Furthermore, although our model may capture genuine preference shifts, it could also be interpreted as a guide for developing more complex preference representations that also abandon the postulate of constant risk aversion.

The rest of the paper is organized as follows. Section 2 develops the state-dependent model and studies the implications for IMRS risk. Asset prices with Markov preferences are discussed in Section 3. The econometric model and estimation results are presented in Section 4. Finally, a conclusion reviews the main findings as well as suggestions for future research.

2 The Model

Consider an economy populated by a large number of identical and infinitely-lived consumers. They choose optimal consumption $C_t$ of the composite nonstorable good, the quantity of stocks $Z_t$ and the quantity of bonds $B_t$ to solve:

$$V(Z_{-1}, B_{-1}) = \max_{(C_t, Z_t, B_t)} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U(C_t, S_t),$$

(2.1)
subject to the following budget constraint:

\[ C_t + P_t Z_t + Q_t B_t \leq W_t + (P_t + D_t)Z_{t-1} + B_{t-1}, \quad (2.2) \]

where \( P_t \) is the price of stock, \( Q_t \) is the price of a real, risk-free, one-period bond, and \( W_t \) is labor income.

Within-period utility \( U(\cdot, S_t) \) is monotone increasing and concave for all stochastic preference states denoted \( S_t \).\(^1\) In particular, we assume that within-period utility is conditionally iso-elastic:

\[ U(C_t, S_t) = \Theta \frac{(\Theta^{-1} C_t)^{1-\gamma_t}}{1-\gamma_t}, \quad \gamma_t = \gamma(S_t) \quad (2.3) \]

where \( \Theta > 0 \) is a constant preference parameter, and where \( \gamma_t \) is a continuous, strictly positive state-dependent curvature index that plays a crucial role in characterizing the agent’s preferences towards risk.

Consider the a-temporal case where the outcome \( l \in L \) is independent of the preference state \( s \in S \), with probabilities given by \( P_l \) and \( \pi_s \), respectively, such that the agent’s preferences are:

\[ V \equiv EU(C, S) = \sum_{l \in L} \sum_{s \in S} P_l \pi_s \Theta \frac{(\Theta^{-1} C_l)^{1-\gamma_s}}{1-\gamma_s}, \quad \gamma_s \geq 0 \quad (2.4) \]

\[ = \sum_{l \in L} P_l U(C_l), \quad \text{where, } U(C_l) = \sum_{s \in S} \pi_s \Theta \frac{(\Theta^{-1} C_l)^{1-\gamma_s}}{1-\gamma_s}. \]

The state-independent local utility function \( U(\cdot) \) is a linear combination with positive weights of conditionally iso-elastic concave functions, and is thus unambiguously concave as well. Moreover, given that \( S \) and \( L \) are orthogonal, the indifference curves will be linear and parallel in the lottery’s probability simplex. As such, the curvature of \( U(\cdot) \) captures the agent’s attitudes toward a-temporal risk; just as in the standard case, concavity of \( U(\cdot) \) implies that a mean-preserving spread reduces the agent’s utility. Consequently, the Arrow-Pratt

\(^1\)Using the method outlined in Constantinides (1982) [Lemma 2, pp. 258–259], it can be shown that if individual sub-utility functions depend (to a degree that varies for each individual) upon a common state, then the “composite” consumer preferences are also state dependent, and given by:

\[ U(C_t, S_t) = \max_{\{C_{it}\}} \sum_{t=1}^{n} \lambda_t U_t(C_{it}, S_t) \]

where \( \lambda \in \mathbb{R}_+^n \) is a vector of weights.
coefficient of relative risk aversion with respect to lotteries on $L$ can be written as:

$$-C \frac{U''(C)}{U'(C)} = \frac{\sum_s \pi_s \gamma_s (\Theta^{-1} C)^{-\gamma_s}}{\sum_s \pi_s (\Theta^{-1} C)^{-\gamma_s}}. \quad (2.5)$$

If preferences are state-independent, i.e. $\gamma_s = \gamma \forall s$, then the coefficient of relative risk aversion is constant and equal to $\gamma$. Furthermore, the coefficient of relative risk aversion for lotteries that are conditional on the realization of a given state $s$ is $\gamma_s$. In our context, each period is associated with a single preference state, so $\gamma_t$ can be interpreted as the coefficient of relative risk aversion for static lotteries.

The specification in (2.3) is a special state-dependent case of the hyperbolic absolute risk aversion (HARA) functionals $U(C) = \Theta(\Theta^{-1} C - \eta)^{1-\gamma}/(1 - \gamma)$, with bliss parameter $\eta$ set to zero. When concavity is time-varying, the parameter $\Theta$ affects marginal rates of substitution. In particular, the effects of fluctuations in concavity on the MRS depend on the level of consumption, relative to $\Theta$, as well as on the covariance between innovations in consumption and curvature. To illustrate, we plot marginal utility $U_c(C_t, S_t)$ for $\gamma_t = \overline{\gamma}$ and for $\gamma_t = \underline{\gamma} < \overline{\gamma}$ in Figure 1.

Figure 1: Concavity and MRS risk

When consumption is low $(C = \underline{C} < \Theta)$, and if risk aversion is counter-cyclical, an unanticipated decline in $C$ could be associated with an increase in $\gamma$ from $\underline{\gamma}$ to $\overline{\gamma}$, which rotates marginal utility clockwise from
\( U_{c}(C, \gamma) \) to \( U_{c}(C, \gamma) \). Similarly, if consumption is high \( (C = \overline{C} > \Theta) \), and if risk aversion is pro-cyclical, an unanticipated decrease in output could be associated with a reduction in \( \gamma \) from \( \gamma \) to \( \gamma \). Both shocks result in much larger fluctuations in MRS’s than a standard, iso-elastic model would suggest – which would correspond to movements in consumption in the vicinity of \( \Theta \). Depending on the direction of co-movements and levels of consumption, state-dependent risk aversion can thus generate greater MRS risk, against which the agent will want to hedge in choosing the optimal portfolio.

Returning to the representative agent’s problem, substitute preferences (2.3) in the program (2.1) to obtain the following Euler equations:

\[
1 = \beta E_t \left[ \frac{(\Theta^{-1}C_{t+1})^{-\gamma_{t+1}}}{(\Theta^{-1}C_t)^{-\gamma_t}} \left( \frac{P_{t+1} + D_{t+1}}{P_t} \right) \right],
\]

(2.6a)

for stocks, while the risk-free asset satisfies:

\[
1 = \beta E_t \left[ \frac{(\Theta^{-1}C_{t+1})^{-\gamma_{t+1}}}{(\Theta^{-1}C_t)^{-\gamma_t}} \left( \frac{1}{Q_t} \right) \right].
\]

(2.6b)

We can further assume that the production technology is homogeneous in capital and labor, and that the net supply of stocks and bonds is one and zero respectively. In this case, since all agents are identical, and the consumption good is nonstorable, this economy will be characterized by a no-trade equilibrium in which the agents hold their asset endowments and consume all output (Abel 1994, Rouwenhorst 1995, Jermann 1998), i.e. \( C_t = D_t + W_t \). Following standard practice in the empirical asset pricing literature, the rest of this study treats consumption as exogenous, and models consumption and dividend growth as two distinct processes.

### 3 Asset Prices and State-Dependent Risk Aversion

Figure 2 plots the fitted asset prices generated by the standard model (dashed line) along the realized series (solid line). It is well known that asset prices are much more volatile than the model would predict (LeRoy
and Porter 1981, Shiller 1981, Cochrane 1992, among others). Furthermore, stock prices remain ‘too high’ or ‘too low’ for periods of several years, episodes that are often referred to as an asset market cycle of bull and bear markets.

Figure 2: Actual and fitted (state-independent preferences) prices

Note: Logarithm of actual S&P composite stock price (solid line) and fitted stock prices (dashed line) for state-independent model \( P_t = B_t C_t; B_t = \beta E_t [B_{t+1} + M_{t+1}]; M_{t+1} = \exp (d_{t+1} - \gamma o_{t+1}). \) Log consumption and dividends follow the VAR(1) process (4.1).

One approach to reproduce the cyclical nature of asset prices is to specify the growth of endowments as a Markov process as in Cecchetti, Lam and Mark (1990, 1993). However, this does not compensate for the fact that consumption and dividend series remain comparatively smooth: predicted asset prices have cycles, but the amplitude of the fitted cycle is much smaller than that of observed asset prices. In any case, our
data are poorly served by a Markov switching model for endowments; a formal model comparison exercise decisively rejects this specification in favor of a VAR alternative.\(^2\)

Alternatively, suppose that risk aversion \(\gamma_t\) follows a discrete-time two-state Markov process:

\[
\gamma_t = \begin{cases} 
\gamma_0 \geq 0, & \text{if } S_t = 0 \\
\gamma_1 \geq 0, & \text{if } S_t = 1 
\end{cases}
\]

where \(P(S_{t+1} = s|S_t = s) = \pi_s, s = 0, 1\). If prolonged periods of under- and over-pricing cannot be explained by regime switches in the endowment process, the question then becomes whether they can be rationalized by applying the Markov process on risk aversion instead.

Next, it can be shown that appropriate restrictions on technological shocks imply that equilibrium dividends and consumption are jointly log-normal (Abel 1994, p. 352):

\[
\begin{bmatrix} d_{t+j} \\ c_{t+j} \end{bmatrix} \sim_t \mathcal{N} \left( \begin{bmatrix} \mu_{d,t+j} \\ \mu_{c,t+j} \end{bmatrix}, \begin{bmatrix} \sigma_{dd,t+j} & \sigma_{dc,t+j} \\ \sigma_{cd,t+j} & \sigma_{cc,t+j} \end{bmatrix} \right), \quad j \geq 1,
\]

where lower-case letters denote logs, and \(\sim_t\) denotes the distribution conditional on information in period \(t\). Given these distributional assumptions, the Appendix demonstrates that state-dependent stock and bond prices are:

\[
P_t^* = B_{p,t}^* \left( \Theta^{-1} C_t \right)^{\gamma_t}, \\
Q_t^* = B_{q,t}^* \left( \Theta^{-1} C_t \right)^{\gamma_t}, \tag{3.3a}
\]

\[
\begin{bmatrix} \Delta c_t \\ \Delta d_t \end{bmatrix} = \begin{bmatrix} \alpha^c_t \\ \alpha^d_t \end{bmatrix} + \left( \begin{bmatrix} \alpha^c_t \\ \alpha^d_t \end{bmatrix} \right) S_t + \left( \begin{bmatrix} \epsilon^c_t \\ \epsilon^d_t \end{bmatrix} \right)
\]

where innovations \(\epsilon_t\) are i.i.d. normal, and \(S_t\) follows a two-states Markov process, against the alternative of the VAR model (4.1). For both models, we used standard reference priors and a “training sample” of 50\% of the data to obtain proper priors in order to compare their predictive power in explaining the latter half of the data. The log Bayes factor was 50 in favor of the VAR, which implies overwhelming rejection of the Markov model. Bonomo and Garcia (1994) also reject the two-state Markov switching model imposed on endowments, when tested against a more general class of Markov structures.
for preference states \( s = 0, 1 \). For stocks, the coefficients \( B_{p,t}^s \) solve

\[
\begin{bmatrix}
    B_{p,t}^0 \\
    B_{p,t}^1
\end{bmatrix} = \beta E_t \begin{bmatrix}
    \pi_0 & 1 - \pi_0 \\
    1 - \pi_1 & \pi_1
\end{bmatrix} \left( \begin{bmatrix}
    B_{p,t+1}^0 \\
    B_{p,t+1}^1
\end{bmatrix} + \begin{bmatrix}
    M_{p,t+1}^0 \\
    M_{p,t+1}^1
\end{bmatrix} \right),
\]

(3.3c)

where

\[
M_{p,t+1}^s \equiv D_{t+1}(\Theta^{-1} C_{t+1})^{-\gamma_s}.
\]

(3.3d)

For the risk-free asset, we obtain:

\[
\begin{bmatrix}
    B_{q,t}^0 \\
    B_{q,t}^1
\end{bmatrix} = \beta E_t \begin{bmatrix}
    \pi_0 & 1 - \pi_0 \\
    1 - \pi_1 & \pi_1
\end{bmatrix} \left( \begin{bmatrix}
    (\Theta^{-1} C_{t+1})^{-\gamma_0} \\
    (\Theta^{-1} C_{t+1})^{-\gamma_1}
\end{bmatrix} \right).
\]

(3.3e)

The transition matrix \( \pi \) and the presence of the subjective discount factor \( \beta \in (0,1) \) guarantee that the loadings \( B_{p,t} \) are bounded if the conditional means of \( M_{p,t} \) do not grow too rapidly. Moreover, the structure can be readily adapted to accommodate time-varying transition matrices \( \pi_t \) or a larger-dimensional state space.

In the standard model, current prices reflect the discounted stream of expected dividends, and the intertemporal marginal rates of substitution between the current period and the period at which dividends are paid out to shareholders serve as the discount factor. If the IMRS are (preference) state-independent, the expression for prices is also state-independent. When Markov preferences are introduced, equations (3.3a)–(3.3d) imply that equilibrium prices must follow:

\[
\begin{bmatrix}
    P_t^0 \\
    P_t^1
\end{bmatrix} = E_t \sum_{j=1}^{\infty} D_{t+j} \begin{bmatrix}
    \psi_{00,j} \text{IMRS}_{t+j}^{0,0} + \psi_{01,j} \text{IMRS}_{t+j}^{0,1} \\
    \psi_{10,j} \text{IMRS}_{t+j}^{1,0} + \psi_{11,j} \text{IMRS}_{t+j}^{1,1}
\end{bmatrix},
\]

(3.4)

where \( \text{IMRS}_{t+j}^{k,l} \equiv \beta^j (\Theta^{-1} C_{t+j})^{-\gamma_l}/(\Theta^{-1} C_t)^{-\gamma_k} \), for \( k, l \in \{0, 1\} \), and \( \psi_{j} \equiv \pi^j = \Pi_{l=1}^j \pi \). Conditional upon the current state, the discount factors are a weighted average of two IMRS. If the elements of the transition matrix \( \pi \) are all strictly positive, all elements of \( \psi_{j} \) will be positive for all \( j \), so the agent must always take
into account the non-zero probability for the event that the preference state that prevails at the purchase of the asset will not be the same as the one that will occur when dividends are paid out.

The implications for predicted stock price volatility are intuitive. If risk aversion is counter-cyclical, if current consumption is low, and if the off-diagonal terms in $\pi$ are non-zero, there is a positive probability that the agent will reap dividends when marginal utility is much lower. As a result, holding stock entails an additional risk over and above any positive covariance between dividends and consumption, and the price he is willing to pay is reduced. Conversely, if high current consumption coincides with a low curvature index, the possibility that the agent may be in a high marginal utility state when dividends are paid out reduces the risk of holding the asset. Consequently, it increases the price he is willing to pay for the stock.

In this framework, a shift from the good state to the bad state, followed by an anticipated return to the good state would signal the beginning of a bear market. This sequence of movements from one preference state to another forms the basis for an explanation for the cyclical movements in stock prices and for the persistent errors of the standard model. Perhaps more significantly, these movements are consistent with rational behavior, and need not be ascribed to some unpleasant modeling defect such as a violation of the transversality condition, or a persistent dichotomy between subjective and objective distributions (Cecchetti, Lam and Mark 1998). Finally, we should emphasize that the regime shifts in the model should not be interpreted as permanent structural breaks for preferences. Indeed, the large cyclical fluctuations in stock prices following a regime shift are generated by the belief that preferences will eventually return to the initial state.

4 Empirical analysis

4.1 Estimation

This section specifies our statistical model, sets out our prior beliefs for the parameters, and estimates the model.
We suppose that the first differences in logs of \( C_t \) and \( D_t \) are fairly well-described by a VAR(1) model:

\[
\begin{bmatrix}
    d_t - d_{t-1} \\
    c_t - c_{t-1}
\end{bmatrix} =
\begin{bmatrix}
    \phi_d \\
    \phi_c
\end{bmatrix} +
\begin{bmatrix}
    \phi_{dd} & \phi_{dc} \\
    \phi_{cd} & \phi_{cc}
\end{bmatrix}
\begin{bmatrix}
    d_{t-1} - d_{t-2} \\
    c_{t-1} - c_{t-2}
\end{bmatrix} +
\begin{bmatrix}
    \varepsilon_{d,t} \\
    \varepsilon_{c,t}
\end{bmatrix},
\]

(4.1)

where \( [\varepsilon_{d,t}, \varepsilon_{c,t}]' \sim iid \ N(0, \Sigma) \). In order to simplify the exposition below, we denote the joint density for a sample of observations generated by (4.1) by \( P(d, c|\phi, \Sigma) \), where the dependence on the pre-sample values of \( d_0 \) and \( c_0 \) has been suppressed, \( d \) and \( c \) represent the vector of \( T \) time series observations for \( d_t \) and \( c_t \), and where \( \phi \equiv [\phi_d, \phi_{dd}, \phi_{dc}, \phi_c, \phi_{cd}, \phi_{cc}]' \). If \( S \) represents the time series of \( T \) realizations for the states, let \( P(S|\pi) \) denote the probability of observing this sequence\(^3\). Given sequences of \( B^*_{p,t} \) and \( B^*_{q,t} \) that satisfy (3.3c) – (3.3c), we model the state-dependent asset prices by

\[
\begin{align*}
p_t &= S_t[b^*_{p,t} + \gamma_1(c_t - \theta)] + (1 - S_t)[b^0_{p,t} + \gamma_0(c_t - \theta)] + u_{p,t} \\
q_t &= S_t[b^*_{q,t} + \gamma_1(c_t - \theta)] + (1 - S_t)[b^0_{q,t} + \gamma_0(c_t - \theta)] + u_{q,t}
\end{align*}
\]

(4.2)

where \( b^*_{p,t} \equiv \log[B^*_{p,t}] \), \( b^*_{q,t} \equiv \log[B^*_{q,t}] \), and where \( u_t \equiv [u_{p,t}, u_{q,t}]' \) is an \( iid \ N(0, \Omega) \) vector of errors generated by our rational expectations equilibrium model.\(^4\) Let \( p \) and \( q \) represent the set of \( T \) observations of \( p_t \) and \( q_t \), respectively. Although the parameters \( \beta, \pi, \phi, \Sigma \) and the dividend series \( d \) do not appear explicitly in (4.2), these values are used to compute \( b^*_{p,t} \) and \( b^*_{q,t} \) [see the Appendix for details]. In order to clarify the role of these parameters and of the dividend series in determining asset prices, we denote the joint density for asset prices by \( P(p, q|d, c, S, \beta, \gamma_0, \gamma_1, \theta, \phi, \Sigma, \pi, \Omega) \).

Taken together, the conditional densities for \( (d, c) \), \( S \) and \( (p, q) \) form a joint density for asset prices, preference states, dividends and consumption that is used as a basis for estimating our model:

\[
P(p, q, d, c, S|\beta, \gamma_0, \gamma_1, \theta, \phi, \Sigma, \pi, \omega) = P(p, q|d, c, S, \beta, \gamma_0, \gamma_1, \theta, \phi, \Sigma, \pi, \Omega)P(d, c|\phi, \Sigma)P(S|\pi)
\]

(4.3)

\(^3\)If \( S_0 \) is the pre-sample observation, then this probability can also be written as \( P(S|\pi, S_0)P(S_0) \). Since the unconditional probability \( P(S_0) \) depends on \( \pi \), we adopt a simpler representation here.

\(^4\)These error terms have no structural interpretation; they simply reflect the inability of this model to produce a perfect fit for observed asset prices throughout the sample.
The structure of our model allows us to make use of a Bayesian MCMC estimation algorithm and to deal with issues such as the unobserved nature of the preference states and the imposition of the regularity conditions required by the structural model. The prior distributions used in the estimation are:

\[
\begin{align*}
P(\beta) &= B(997.5, 2.5) \quad P(\gamma_0, \gamma_1) = N_+(\mu_2, 100^2 I_2) \quad P(\theta) = N(0, 300^2) \quad P(\pi_s) = B(95, 5), \ s = 0, 1 \\
P(\phi) &= N(0_6, 10^2 I_6) \quad P(\Sigma^{-1}) = W(0.025^2 I_2, 5) \quad P(\Omega^{-1}) = W(0.025^2 I_2, 5)
\end{align*}
\]

(4.4)

where \(B\), \(W\) and \(G\) respectively denote the beta, Wishart and gamma distributions, and where \(\iota\) is the unit vector.

Although all prior distributions are proper, most are quite diffuse; the priors for \(\gamma_0, \gamma_1, \theta, \phi\) are all locally uniform in the region in which the likelihood function has mass. The Wishart prior distributions for \(\Sigma^{-1}\) and \(\Omega^{-1}\) are centered over \(\sigma_d = \sigma_c = \omega_p = \omega_q \approx 0.01\) and \(\sigma_{dc} = \omega_{pq} = 0\), but setting the degrees of freedom parameter at 5 means that the prior will play only a small role in comparison to the information contained in the 390 observations of our sample.

Stronger priors are adopted for the subjective rate of discount and the transition probabilities. The prior for \(\beta\) is consistent with a mean annual rate of time preference of 0.03 and a prior standard deviation of about 0.02. The prior probability of staying in a given state is represented by a beta distribution that is consistent with a mean prior expected duration of roughly two years for each state, and with a prior standard deviation of 16 months; the size of the fictitious prior sample is 100, which is roughly one-fourth the size of the sample. The \(iid\) nature of the priors for the transition probabilities implies that the prior unconditional probability of being in the good state is 0.5. The conditions \(0 < \beta < 1\) and \(\gamma_0, \gamma_1 > 0\) ensure the existence of a general equilibrium, and are imposed by the priors. The additional restriction \((\gamma_1 - \gamma_0)(c_1 - \theta) > 0, \forall t\) identifies \(S_t = 1\) as the state with lower marginal utility of consumption (good state).
4.2 Results

The Markov model of preference regimes is estimated using US monthly series for stock prices and dividends (S&P composite stock indices); aggregate consumption (real per-capita expenditures on nondurables and services), and bonds (3-months T-Bills). The sampling period is 1960:2 to 1992:9 (390 observations).

The complex nature of the data density (4.3) and the fact that we do not observe the latent state variable $S$ poses significant problems for classical estimation of our model. For example, Cecchetti, Lam and Mark (1993) are obliged to make use of a mixture of estimation and calibration techniques in order to apply their model to data, instead of their preferred, likelihood-based approach (Cecchetti et al. 1993, p.30, fn. 12).

The development of Bayesian MCMC and data augmentation techniques (Chib and Greenberg 1996, provide a survey) permits us to make exact (finite-sample) inferences based on the exact likelihood function associated with (4.3). In the Markov chain used here (see the Appendix for details), we make use of the Gibbs sampler and data augmentation to generate a sequence of autocorrelated draws from the joint posterior distribution

$$P(\beta, \gamma_0, \gamma_1, \theta, \phi, \Sigma, \pi, \omega, S|p, q, d, c).$$

These sequences are used to compute the posterior moments of the various parameters reported in Table 1 and to estimate the posterior probabilities that $S_t = 1$ at each data point.

The risk aversion coefficients $\gamma_0$ and $\gamma_1$ – for which weak priors were used – are well within the range that is usually considered reasonable. Moreover, our estimates suggest that risk aversion is counter-cyclical: the state with high asset prices is also associated with low risk aversion. In Section 2, it was noted that this would be true for values of $\Theta^{-1}C$ that are less than one. In our sample, $\log[C]$ runs from 8.0 to 8.4, so our estimate for $\theta \equiv \log[\Theta]$ is consistent with this interpretation. Although the difference between $\gamma_0$ and $\gamma_1$ does not appear to be large, the implications for marginal utility are quite dramatic: the transition from the good state to the bad state increases marginal utility by about 50%. The posterior mean of the discount factor $\beta$ – for which we used fairly strong priors – corresponds to an annual discount rate of 3.1%.
Table 1: Posterior Moments: Markov Preferences Model

<table>
<thead>
<tr>
<th>parameter</th>
<th>mean</th>
<th>std. deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta$</td>
<td>0.9974</td>
<td>0.0001</td>
</tr>
<tr>
<td>$\gamma_0$</td>
<td>2.6513</td>
<td>0.0650</td>
</tr>
<tr>
<td>$\gamma_1$</td>
<td>2.2059</td>
<td>0.0964</td>
</tr>
<tr>
<td>$[\gamma_0 - \gamma_1]$</td>
<td>0.4454</td>
<td>0.0874</td>
</tr>
<tr>
<td>$\theta$</td>
<td>9.2269</td>
<td>0.3022</td>
</tr>
<tr>
<td>$\phi_d$</td>
<td>-2.65e-04</td>
<td>2.39e-04</td>
</tr>
<tr>
<td>$\phi_c$</td>
<td>2.65e-04</td>
<td>0.90e-04</td>
</tr>
<tr>
<td>$\phi_{dd}$</td>
<td>0.0026</td>
<td>0.0357</td>
</tr>
<tr>
<td>$\phi_{dc}$</td>
<td>0.0661</td>
<td>0.0225</td>
</tr>
<tr>
<td>$\phi_{cd}$</td>
<td>0.0658</td>
<td>0.0115</td>
</tr>
<tr>
<td>$\phi_{cc}$</td>
<td>-0.0206</td>
<td>0.0157</td>
</tr>
<tr>
<td>$\Sigma_{dd}$</td>
<td>8.32e-05</td>
<td>0.60e-05</td>
</tr>
<tr>
<td>$\Sigma_{dc}$</td>
<td>7.14e-06</td>
<td>3.58e-06</td>
</tr>
<tr>
<td>$\Sigma_{cc}$</td>
<td>4.49e-05</td>
<td>0.45e-05</td>
</tr>
<tr>
<td>$\sigma_d$</td>
<td>0.0091</td>
<td>0.0003</td>
</tr>
<tr>
<td>$\rho_{dc}$</td>
<td>0.1006</td>
<td>0.0493</td>
</tr>
<tr>
<td>$\sigma_c$</td>
<td>0.0078</td>
<td>0.0003</td>
</tr>
<tr>
<td>$\pi_0$</td>
<td>0.9909</td>
<td>0.0004</td>
</tr>
<tr>
<td>$\pi_1$</td>
<td>0.9939</td>
<td>0.0004</td>
</tr>
<tr>
<td>$\Omega_{pp}$</td>
<td>0.0108</td>
<td>0.0008</td>
</tr>
<tr>
<td>$\Omega_{pq}$</td>
<td>-5.2e-05</td>
<td>3.18e-05</td>
</tr>
<tr>
<td>$\Omega_{qq}$</td>
<td>2.78e-05</td>
<td>2.16e-06</td>
</tr>
<tr>
<td>$\omega_p$</td>
<td>0.1039</td>
<td>0.0040</td>
</tr>
<tr>
<td>$\rho_{pq}$</td>
<td>-0.0949</td>
<td>0.0573</td>
</tr>
<tr>
<td>$\omega_q$</td>
<td>0.0033</td>
<td>0.0002</td>
</tr>
</tbody>
</table>

Note: posterior means and standard deviations for parameters of Markov preferences model (4.3).

Both preference states are quite persistent, with a slightly higher probability of staying in state 1. The mean expected duration of a bear market $S_t = 0$ is 9.1 years, while that of the bull market is roughly 13.6 years. The unconditional probability that a bull market will prevail in a given month is 0.60, and our estimate for the sample mean of $S_t$ is 0.69. Although our prior means for both $\pi_0$ and $\pi_1$ are set at the fairly large value of 0.95, the posterior means are even larger, suggesting that the high estimated persistence is not an artifact of our priors; the data have the effect of revising upwards the probability of staying in a given state. If weaker priors are used, these probabilities are revised even further upwards.

Figure 3 plots observed prices with the fitted prices generated by the Markov preferences model: the improvement over the fit of the fixed preference model in Figure 2 is striking. The square root of the mean...
Figure 3: Actual and fitted (Markov preferences) prices

Note: Logarithm of actual S&P composite stock price (solid line) and fitted stock prices (dashed line) for Markov preferences model (4.1) and (4.2).

The squared error (RMSE) of the fitted log stock price series is reduced from 0.21 to 0.10, and the coefficient of correlation between the fitted and actual series increases from 0.65 to 0.93. An interesting feature of our model is that within a given phase, its predictions look much like those of the standard model; the improvements are almost entirely due to the model’s ability to track the sharp price movements around the oil price shock and at the beginning of the bull market in the late 1980’s.

Even though our model is able to reproduce the strong cyclical movements in stock prices, it is also able to capture the weak variability of bond prices as well. The mean of the fitted log bond price series (-0.0018) is within the range of the observed value (-0.0014), and the standard deviation of the predicted series (0.0032) is close to that of the observed series (0.0039). The RMSE of the fitted log bond price series is 0.0044, and the coefficient of correlation between the actual and fitted series is 0.26.
Figure 4 plots the posterior probability that a given observation occurred during a bull market. We noted earlier that our results suggest that bear markets prevailed for roughly a third of our sample; Figure 4 indicates there were two or perhaps three bear markets during the period 1960:3–1992:9. The first episode occurs during the four-month period 1970:5–1970:8, when there is a sharp drop in the bull market probability, although the fact that the lowest probability is 0.79 suggests that this period is best classified as a bull market, albeit a weak one. After that, the first clear bear market starts in 1974:7 and ends 17 months later in 1975:12, while the last bear market of the sample begins in 1977:4, lasting for over 8 years before ending in 1985:11.

It is also interesting to compare our financial cycle chronology with the University of Michigan index of consumer expectations (dashed line) and the NBER reference business cycle recessionary periods (shaded regions). In the first half of the sample, the three series display a high degree of coherence, but from 1976 on,
the relationship between the three indices appears to weaken: the bear market of 1977-85 persists through two recoveries, and the bull market that began in 1985 continues right through the sharp recession of 1990-91 and on to the end of the sample. Since the asset price series in Figures 2 and 3 remains low until 1985 and does not appear to be greatly affected by the 1990-91 recession, this puzzle does not indicate an obvious problem with the model’s ability to explain the data.

4.3 A Digression on Asset Returns Puzzles

We have shown that the two-state Markov preferences model performs surprisingly well in capturing the cyclical components in stock and bond prices. We conclude this study by a brief discussion of the implications of state-dependent risk aversion for the empirical puzzles discussed earlier (Gordon and St-Amour 1998, provide a detailed treatment).

Consider the continuous-time setting and let \( r_p(t) \) and \( r_q(t) \) denote the net rate of returns on stock and bonds respectively. Furthermore, assume that \( c(t) \), \( \gamma(t) \), and \( r_p(t) \) follow joint Itô processes.⁵ Then, it can be shown that (2.6a) and (2.6b) imply that mean excess returns and the risk-free rate must satisfy:

\[
E(t)[r_p(t) - r_q(t)] = [c(t) - \theta]\sigma_{cp}(t) + \gamma(t)\sigma_{cp}(t) - 0.5\sigma_{pp}(t) \tag{4.6a}
\]

where \( \theta \equiv \log[\Theta] \) and

\[
r_q(t) = -\log(\beta)dt - E(t)[du_c(t)] - 0.5\sigma_{uu}(t)
\]

\[
= -\log(\beta)dt + \gamma(t)E(t)[dc(t)] + [c(t) - \theta]E(t)[d\gamma(t)] + \{1 - \gamma(t)[c(t) - \theta]\}\sigma_{\gamma}(t)
\]

\[
-0.5\gamma(t)^2\sigma_{cc}(t) - 0.5[c(t) - \theta]^2\sigma_{\gamma\gamma}(t) \tag{4.6b}
\]

where \( \sigma_{ij}(t) \) denotes the covariance between innovations in elements \( i, j \in \{u, c, \gamma, p\} \).

⁵The assumption that \( \gamma(t) \) follows an Itô process can be justified if our model is considered as a proxy for wealth-dependent preferences similar to Bakshi and Chen (1996), in which optimal consumption remains proportional to wealth. Gordon and St-Amour (1998) show that changes in curvature can capture changes in log wealth. Hence, the assumption that net returns follows a Brownian motion extends to changes in log consumption and changes in \( \gamma(t) \) as well.
It is typically observed that $\sigma_{cp}(t) > 0$, i.e. low returns are associated with adverse consumption innovations. For high (low) marginal utility, if returns and risk aversion are negatively (positively) correlated, then $[c(t) - \theta]\sigma_{cp}(t) > 0$ in (4.6a). Hence, holding the risky asset entails a larger risk to marginal utility - for reasons discussed in Section 2 - due to the coincidence of low returns, low consumption and high risk aversion. This second risk justifies the high observed premia without having to inflate risk aversion to unrealistic levels.

By a similar reasoning, the concavity variance term $\sigma_{\gamma\gamma}(t)$ in (4.6b) increases the volatility of the MRS. Moreover, the term $\{1 - \gamma(t)[c(t) - \theta]\} \sigma_{\gamma c}(t)$ is negative when consumption is low (high) and risk aversion is counter- (pro-) cyclical. Both elements induce a precautionary demand for the risk-free asset and therefore reduce its return.

Finally, even if innovations are homoscedastic, i.e. $\sigma_{ij}(t) = \sigma_{ij}$, for $i, j \in \{c, \gamma, p\}$, and $\forall t$, the risk premium is time-varying, since log consumption is the loading for concavity risk. If risk preferences are not constant, then ARCH-type structures are not necessarily required in order to capture cyclical movements in risk premia. If risk aversion is counter-cyclical ($\sigma_{\gamma p} < 0$) and consumption is low ($c < \theta$), recessions would be associated with increasing risk premia as consumption falls and risk aversion increases, while expansions would witness a decline in excess returns.

The extra source of risk in the premium (4.6a) is the distinguishing feature of our state-dependent risk preference specification. The preference state variable used in Campbell and Cochrane (1995) is a deterministic function of consumption, so all IMRS risk must be explained by the consumption covariances. Not allowing any independent sources of IMRS risk implies that the slow-moving habit model has difficulty to explain observed equity premia with reasonable levels of relative risk aversion.

We can estimate the expression for excess returns (4.6a) using exact likelihood methods for continuous-time models outlined in Gordon and St-Amour (1998). Our estimates for the risk aversion series for this data set fluctuate in a narrow band around a sample mean of 0.80, and our estimate for the sample standard

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6The non-linearity in the process for bond returns implies that exact likelihood approaches are difficult to use for (4.6b). Full details and estimation results can be obtained from the authors upon request.
deviation is 0.38. Both values suggest again that comparatively small movements in risk aversion are able to generate enough movement in the IMRS to explain asset market behavior using plausible levels of risk aversion. Since the posterior probability that \( [c(t) - \theta] \sigma_r > 0 \) is greater than 0.98, the model is able to fit excess returns with reasonable values of risk aversion. Indeed, since the consumption risk term is only 1% of the concavity risk term, it is perhaps unsurprising that the standard model fares so poorly.

Although the subjective scaling parameter \( \theta \) is not identifiable from excess returns alone, we conducted an *ad hoc* calibration exercise in which \( c(t) \) and \( \gamma(t) \) were approximated by their time-averaged values, and the parameter estimates were used to compute the various conditional expectations in (4.6b). Two free parameters remain, so we set \( \beta = 1.03^{-1/12} \), and searched for a value of \( \theta \) so that the sample mean of the fitted risk-free rate matches that of the data. This exercise produces a value of \( \theta = 8.8 \), which is remarkably close to the estimated value of 9.23 obtained in the Markov preference model.

The Itô model produces a great deal more volatility in preferences toward risk than the Markov model and this illustrates some of the limitations of a framework in which preference states are modeled as latent variables. On the other hand, since our model appears to go some way in explaining the puzzles associated with specifications in which risk preferences are fixed, our results can be used as a guide in developing functional forms in which risk preferences are allowed to be time-varying.

5 Conclusion

This study shows that the addition of state-dependent risk preferences to the original C-CAPM model is able to capture many of the salient features of financial market data. The gains stem chiefly from the additional volatility to the IMRS that state dependencies are able to provide.

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7It should be kept in mind that the range of fluctuations that we find is in fact significantly smaller in magnitude than those obtained in other contexts in which risk aversion is time-varying. For example, Chou, Engle and Kane (1992), p. 206 using rolling regression techniques with similar data, find a time-varying price of risk over weekly series that ranges from -0.4 to 15.6, with a mean of 5.4, and a standard deviation of 4.1. When we use the calibrated parameters of Campbell and Cochrane (1995) in their slow-moving habit model applied to our data set, the induced risk aversion parameter ranges from 21 to 81, with a mean of 34, and a standard deviation of 10.6.
We study stock and bond prices using a simple two-state Markov process for the parameter associated with risk aversion. Our estimates indicate that small but significant movements in the agent’s concavity index are able to generate the sharp swings in asset prices that are characteristic of financial market data. These shifts are infrequent, and both states display a high degree of persistence. For much of the sample, our estimated chronology of bull and bear markets displays interesting parallels with consumer expectations indices and with the business cycle. A noteworthy finding is that in the latter part of our sample (1977–1992) there is less coherence between the bull and bear markets of the financial market and the expansions and recessions of the goods market.

This paper is more descriptive than explanatory, and it raises perhaps more questions than it answers. For example, how and why the financial cycle lost much of its coherence with the business cycle in the mid-1970’s is an empirical puzzle that should be addressed in future research. Furthermore, this study is agnostic about why preferences might shift: potential explanations include movements in wealth or shifts in relative prices. Future work should address the issue of determining the factors that underlie the movements in risk preferences that are identified here.
A Derivation of the state of Markov preferences pricing kernels

A given realization of the state variable $S_t$ affects equilibrium asset prices through the (quasi reduced-form) demand schedules $P_t = P(C_t, B_t, S_t)$; we denote $P^*_t$ as the observed equilibrium price for asset $i$ given the realized state $S_t = s \in \{0,1\}$. Substituting the transition probabilities in the Euler equations reveals that risky asset prices must satisfy:

$$
1 = \beta \left\{ \pi_0 E_t \left[ \frac{(P^0_{t+1} + D_{t+1})}{P_t} \right] \left( \frac{\bar{\theta}^{-1}(C_{t+1})^{-1}}{\bar{\theta}(\bar{C}_{t+1})^{-1}} \right) + (1 - \pi_0) E_t \left[ \left( \frac{(P^0_{t+1} + D_{t+1})}{P_t} \right) \left( \frac{\bar{\theta}^{-1}(C_{t+1})^{-1}}{\bar{\theta}(\bar{C}_{t+1})^{-1}} \right) \right] \right\},
$$

(A.1)

while bond prices are obtained by:

$$
1 = \beta \left\{ \pi_0 E_t \left[ \left( \frac{1}{P_t} \right) \left( \frac{\bar{\theta}^{-1}(C_{t+1})^{-1}}{\bar{\theta}(\bar{C}_{t+1})^{-1}} \right) + (1 - \pi_0) E_t \left[ \left( \frac{1}{P_t} \right) \left( \frac{\bar{\theta}^{-1}(C_{t+1})^{-1}}{\bar{\theta}(\bar{C}_{t+1})^{-1}} \right) \right] \right\}.
$$

(A.2)

Substitute the candidate solutions $P^*_t = B_{s,t}^*(\bar{\theta}^{-1}C_t)^{\gamma_t}$ and $Q_t^* = B_{s,t}^*(\bar{\theta}^{-1}C_t)^{\gamma_t}$ in equations (A.1)-(A.2), and use the log-normal properties to solve for $B_{p,t}^*$ and $B_{q,t}^*$ as in (3.3c)-(3.3e).

B Computing Equilibrium Prices

The solution to the difference equation (3.3c) can be written as

$$
\begin{bmatrix}
P_{p,t}^* \\
B_{p,t}^*
\end{bmatrix}
= E_t \sum_{j=1}^{K-1} \beta^j \pi^j \begin{bmatrix}
M_{p,t+j}^0 \\
M_{p,t+j}^1
\end{bmatrix}
= E_t \sum_{j=K}^{\infty} \beta^j \pi^j \begin{bmatrix}
M_{p,t+j}^0 \\
M_{p,t+j}^1
\end{bmatrix}
$$

(B.1)

where $M_{p,t+j} = D_{t+j}(\bar{\theta}C_{t+j})^{-\gamma_t}$ as in (3.3d). For a given value of $K$, we can compute $\{\mu_{d,t+j}, \mu_{c,t+j}\}_{j=1}^K$, the sequence of conditional means for $(d_{t+j}, c_{t+j})$ as well as $\{\sigma_{d,d,t+j}, \sigma_{d,c,t+j}, \sigma_{c,c,t+j}\}_{j=1}^K$, the sequence of conditional variances and covariances, for each $j$ by repeated application of the VAR model in (4.1) for dividend and consumption growth rates in order to compute the partial sum $E_t \sum_{j=1}^{K-1} \beta^j \pi^j M_{p,t+j}$.

The remainder term is not so easily dealt with. One approach would be to set $K$ to be large enough so that $E_t \sum_{j=K}^{\infty} \beta^j \pi^j M_{p,t+j} \approx 0$, but since $\beta$ is close to one and since dividends and consumption grow over time, $K$ must be set to an extremely large value. Since (B.1) must be evaluated at each evaluation of the likelihood, this technique does not appear to be feasible given our current computing facilities.

Instead of choosing $K$ such that the remainder term is zero, we set $K$ so that $\pi^{K+j} \approx \tilde{\pi}$, $E_t M_{p,t+K+j}^0 \approx E_t M_{p,t+K+K} \approx E_t \left[ \sum_{j=K}^{\infty} \beta^j \pi^j M_{p,t+j} \right] \exp \{ j[k_{d} - \gamma_{d} \tilde{\pi} + 0.5(\tilde{\sigma}_{dd} - 2\tilde{\theta}d_{d} + \tilde{\gamma}_{d}^{2}\tilde{\sigma}_{cc})] \}$ for $j = 0, 1, 2, \ldots$ Since $\pi$ is the transition matrix of an ergodic Markov chain, $\pi^j$ converges to a constant matrix of the unconditional state probabilities. If the VAR process in (4.1) is stationary (a condition we impose in the estimation, even though this constraint does not appear to be binding), then period $t$ means for the log differences for dividends and consumption will converge to the unconditional expectations $\bar{\mu}$ and $\bar{\mu}$, and the covariances will converge to $\bar{\sigma}_{dd}$, $\bar{\sigma}_{dc}$ and $\bar{\sigma}_{cc}$.

The appropriate choice for $K$ depends on the desired level of precision and the rate of convergence of the VAR and Markov state processes. Let $\lambda_2$ denote the second-largest eigenvalue (since the process is ergodic, the largest eigenvalue is equal to one) of $\pi$, and let $\lambda_1$ and $\lambda_2$ denote the eigenvalues of the matrix of autocorrelation coefficients in (4.1), and let $\lambda \equiv \max \{ |\lambda_1|, |\lambda_2| \}$. Given the precision of our data (three or four significant digits), $K$ is set so that $\lambda^K < 0.0001$. In the estimation, $K$ is reset to satisfy this condition at every evaluation of the likelihood.

---

8We assume that the relevant transversality condition rules out bubbles.
Given $K$ and the growth rates, the terms in the remainder are simply the sums of geometric series. Adding this term to the partial sum yields the value of $B_{p,t}^*$ used in the estimation.

C Estimation Details

Estimates for the model (4.3) are based on draws generated by iterating through the following sequence, which combines the Gibbs sampler with data augmentation:

\begin{alignat}{2}
\beta^i &\sim \mathcal{P}(\beta|\gamma_{0i}^{-1}, \gamma_{1i}^{-1}, \theta^{-1}, \phi^{i-1}, \Sigma^{-1}, \pi^{-1}, \Omega^{-1}, S^{-1}, p, q, d, c) \tag{C.1} \\
(\gamma_{0i}, \gamma_{1i}, \theta^i, \phi^i) &\sim \mathcal{P}(\gamma_{0i}, \gamma_{1i}, \theta^i, \phi^i|\beta^i, \Sigma^{-1}, \pi^{-1}, \Omega^{-1}, S^{-1}, p, q, d, c) \tag{C.2} \\
\phi^i &\sim \mathcal{P}(\phi|\beta^i, \gamma_{0i}, \gamma_{1i}, \theta^i, \Sigma^{-1}, \pi^{-1}, \Omega^{-1}, S^{-1}, p, q, d, c) \tag{C.3} \\
\Sigma^i &\sim \mathcal{P}(\Sigma|\beta^i, \gamma_{0i}^{-1}, \gamma_{1i}^{-1}, \theta^i, \phi^i, \pi^{-1}, \Omega^{-1}, S^{-1}, p, q, d, c) \tag{C.4} \\
\pi^i &\sim \mathcal{P}(\pi|\beta^i, \gamma_{0i}^{-1}, \gamma_{1i}^{-1}, \theta^i, \phi^i, \Sigma^i, \Omega^{-1}, S^{-1}, p, q, d, c) \tag{C.5} \\
\Omega^i &\sim \mathcal{P}(\Omega|\beta^i, \gamma_{0i}^{-1}, \gamma_{1i}^{-1}, \theta^i, \phi^i, \Sigma^i, \pi^i, S^{-1}, p, q, d, c) \tag{C.6} \\
S^i &\sim \mathcal{P}(S|\beta^i, \gamma_{0i}^{-1}, \gamma_{1i}^{-1}, \theta^i, \phi^i, \Sigma^i, \pi^i, \Omega^i, p, q, d, c). \tag{C.7}
\end{alignat}

In the Gibbs sampler steps (C.1)-(C.6), parameter values are simulated from their “full conditional” distributions, i.e., conditional on the observed data, the states $S$ and the other parameters. The data augmentation step (C.7) is the key element of the algorithm, since it avoids the computational burden of direct evaluation of the likelihood function, which would involve integrating $S$ out of the data density (4.3).

Under fairly weak conditions that are satisfied in this application, the sequence of draws generated according to (C.1)-(C.7) forms an ergodic Markov chain whose stable distribution is the joint posterior distribution (4.5). Once the chain has converged to its stable distribution, it produces a sequence of correlated draws from (4.5). Given a sufficiently large number of draws, the posterior moments of the various parameters can be estimated with an arbitrarily high degree of accuracy.

It is possible to simulate draws for two parameters using standard techniques. Since $\Omega$ appears only in (4.2), its full conditional distribution can be derived using well-known results for the bivariate normal model with known variance and a conjugate inverse-Wishart prior. If the rest of the parameters and the states are known, we can recover the series $u_{p,t}$ and $u_{q,t}$ from (4.2). Given these values and the inverse-Wishart prior, the full conditional posterior for $\Omega^{-1}$ is also an inverse-Wishart density; see Poirier (1995, p. 300).

C.1 “Metropolis-within-Gibbs”

Estimation of the other parameters is complicated by the fact that they appear in two of the three data densities that form (4.3). Although direct draws are not available, it is possible to implement a fairly simple Metropolis-Hastings algorithm (MHA) by exploiting the structure of our model.

It is perhaps useful to provide a brief summary of the MHA; a more detailed survey is provided by Chib and Greenberg (1995). Suppose that we wish to simulate draws from a “target density” $f(\lambda)$, and define $q(\lambda, \lambda')$ to be a known density from which it is easy to simulate a candidate $\lambda'$ given $\lambda$, the value generated by the previous iteration if the algorithm. Define

$$
\alpha(\lambda, \lambda') \equiv \max \left[ \frac{f(\lambda')q(\lambda', \lambda)}{f(\lambda)q(\lambda, \lambda')} \right]. \tag{C.8}
$$

The MHA returns the candidate $\lambda'$ with probability $\alpha$; if $\lambda'$ is rejected, then the MHA returns $\lambda$. Typically, some extra work is required in order to identify suitable forms for $q(\lambda, \lambda')$; see Chib and Greenberg (1995).
In estimating our model, we are able to make use of the special case in which the target density can be written as the product of two densities \( g \) and \( h \), so that \( f(\lambda) = g(\lambda)h(\lambda) \). Suppose further that we can simulate draws from \( h \), and set \( q(\lambda, \lambda') = h(\lambda') \). Substituting into (C.8) yields

\[
\alpha(\lambda, \lambda') \equiv \max \left[ \frac{g(\lambda')}{g(\lambda)}, 1 \right]
\]

Here, \( h \) is used to generate new candidates, and \( g \) is used to decide whether or not they are to be accepted.

In our application, we make use of this special case in simulating draws for \( \beta, \phi, \Sigma \) and \( \pi \):

\[\begin{align*}
g(\beta) & : \text{Proportional to } P(p, q|d, c, S, \beta, \gamma_0, \gamma_1, \phi, \Sigma, \pi, \Omega) \text{ considered as a function of } \beta \\
note{\beta} & : \text{appears only in (4.2), and the tight prior for } \beta \text{ plays a relatively important role in the posterior.} \\
h(\beta) & : P(\beta) \\
\phi & : \text{The full conditional distribution of } \phi \text{ is proportional to the product of the prior and the data densities } P(p, q|d, c, S, \beta, \gamma_0, \gamma_1, \phi, \Sigma, \pi, \Omega) \text{ and } P(d|c, \phi, \Sigma). \text{ Given } \Sigma, \text{ the factor } h(\phi) \text{ is simply the posterior for the slope parameters of a SUR model with a known covariance matrix. Chib and Greenberg (1996, section 3.1) note that } h(\phi) \text{ is a multivariate normal, and they provide its mean and variance.} \\
g(\phi) & : \text{Proportional to } P(p, q|d, c, S, \beta, \gamma_0, \gamma_1, \phi, \Sigma, \pi, \Omega) \text{ considered as a function of } \phi \\
\Sigma & : \text{Proportional to } P(p, q|d, c, S, \beta, \gamma_0, \gamma_1, \phi, \Sigma, \pi, \Omega) \text{ considered as a function of } \Sigma \\
\note{\Sigma} & : \text{As was the case for } \phi, \text{ } h(\phi) \text{ is the posterior for the covariance matrix of a SUR model with a known vector of slope parameters. This distribution has an inverse-Wishart form; see Chib and Greenberg (1996, section 3.1) for details.} \\
g(\pi) & : \text{Proportional to } P(p, q|d, c, S, \beta, \gamma_0, \gamma_1, \phi, \Sigma, \pi, \Omega) \text{ considered as a function of } \pi \\
h(\pi) & : P(\pi|S) \propto P(S|\pi)P(\pi) \\
\note{\pi} & : \text{The full conditional distribution of } \pi \text{ is proportional to the product of the prior and the data densities } P(p, q|d, c, S, \beta, \gamma_0, \gamma_1, \phi, \Sigma, \pi, \Omega) \text{ and } P(S|\pi, S_0). \text{ Given } S \text{ and the beta priors for } \pi_0 \text{ and } \pi_1, \text{ } P(\pi|S) \text{ are also beta distributions; see Albert and Chib (1993) for details.}
\end{align*}\]

For \( \beta, \phi, \Sigma \) and \( \pi \), candidates are generated from their various \( h \) densities, and the criterion for deciding whether or not the candidate is accepted is whether or not the new candidate helps fit the asset price series.

Similar intuition is used for the draws for \( \gamma_0 \) and \( \gamma_1 \). New candidates for \( (\log(\gamma_0), \log(\gamma_1), \theta) \) were generated using a normal distribution with means set equal to their previous values; only candidates that satisfy \((\gamma_1 - \gamma_0)/(c_i - \theta) > 0 \) were generated by the algorithm. In this case, the decision rule for accepting the new candidates is also (C.9); see Chib and Greenberg (1995).

C.2 Data Augmentation

As we noted in Section 3.3, simulating the latent variable \( S \) in (C.7) allows us to estimate the model without direct evaluation of the likelihood function. Albert and Chib (1993) note that the full conditional distribution for a given \( S_t \) conditional on the data, the other parameters and the values of \( S \) other than that in period \( t \) - denoted \( S_{-1} \) - is obtained by applying Bayes’ rule:
\[ P(S_t = 1|\beta, \gamma_0, \gamma_1, \theta, \phi, \Sigma, \Omega, S_{t-1}, p, q, d, c) = \frac{P(S_t = 1|S_{t-1})P^*(p_t, q_t|S_t = 1)P(S_{t+1}|S_t = 1)P^*(p_t, q_t|S_t = 0)P(S_{t+1}|S_t = 0)}{P^*(p_t, q_t|S_t = 1)P^*(p_t, q_t|S_t = 0)P^*(p_t, q_t|S_t = 0)P(S_{t+1}|S_t = 0)} \]  

where \( P^*(p_t, q_t|S_t = 0) \) and \( P^*(p_t, q_t|S_t = 1) \) are the densities for the observed \((p_t, q_t)\) pair, given the rest of the model and evaluated at each preference state. Given values for \( S_{t-1} \) and \( S_{t+1} \), the terms \( P(S_t = 1|S_{t-1}) \), \( P(S_{t+1}|S_t = 1) \), \( P(S_t = 0|S_{t-1}) \) and \( P(S_{t+1}|S_t = 0) \) are the appropriate elements of the transition matrix \( \pi \). Values of \( S_T \) and \( S_0 \) were simulated using the relevant adaptations of the expressions in Albert and Chib (1993).
References


