MALTHUSIAN TRAP AND ENDOGENOUS POPULATION

by

Nguyen Manh Hung

and

Paul Makdissi

Cahier 9705

Cahier 97-01
du Groupe de Recherche en Économie de l’Énergie
de l’Environnement et des Ressources naturelles
GREEN

February 1997

DÉPARTEMENT D’ÉCONOMIQUE
Faculté des Sciences Sociales

We thank S. Gordon and especially Y. Richelle for many useful comments. The first author benefits from many discussions with J.P. Amigues and P. Michel and acknowledges a financial support of the SSHRC.
1. Introduction

Recent resurrected interests in the theoretical population problem point to the important connection between economic growth and the fertility decision in the new household economics. In this vein, the pessimistic vision set forth two centuries ago in Malthus’s "Essays in the Principle of Population" comes back to haunt the intellectual debate on the thorny subject of economic sustainability. On the other hand, many theorists - some of them the most prominent - in the neo-classical tradition would have just given to this vision little if any importance. Population growth remains unbounded in Barro and Becker (1989) where fertility decisions are endogenous but labor can be substituted by physical capital that can be accumulated over time. The same conclusion is reached when instead the human capital plays the role of factor substitution in Becker, Murphy and Tamura (1992).

What should then happen if labor substitution is not possible? Does the vision - sometime called the Malthusian trap which pretends that even in the presence of technological progress population growth would ultimately equilibrate the per capita consumption to the level of subsistence? And if so, are there some means to escape from it? This paper addresses to these questions.

In what follows, we consider human being as both the sole asset and a production factor which can be combined with a fixed factor say land to produce a homogeneous commodity. Saving so to speak can only be made through having children the number of which is an endogenous decision to the household. The utility of the representative household takes the "dynastic" form so that the decentralized-decision making of all agents of different generations can be subsumed to the planning Ramsey problem. This problem will thoroughly be studied under both deterministic and stochastic framework.

The model in this paper is put forth in the next Section where we would like to emphasize our "capital" is of a rather special type. In Section 3 the optimal solution to our dynamic planning problem is carried out in some details with particular attention given to the Malthusian equilibrium and the Malthusian trap. In Section 4 the process of population growth is assumed to incorporate a white noise Brownian motion and the saving made through the number of offsprings contains a precautionary component. In the final section we sum up our main results and offer some generalization whenever possible. We also comment on what should next be done as well as some feasible avenues for further research efforts.
2. The Model

The model considered in this paper is a variant of the Solow-Swan model of economic growth where, instead of physical assets, "people" constitutes the only form of productive "capital". In discrete time, each individual agent living for 1 period earns an income $y_t$ has to make decision about his consumption $c_t$, the number of his offspring $n_t$, who become themselves adults at $t+1$. Suppose that the utility of each agent is a function of its own consumption $U(c_t)$ and the utility of all of his descendants:

$$V_t = U(c_t) + n_t \delta \alpha (n_t) V_{t+1},$$

where $\delta$ denotes the discount factor of time preference and where $\alpha (n)$ measures the degree of altruism of the parent toward each child. This degree $\alpha$ is likely a decreasing function of $n_t$, but following Becker and Barro (1988) $\Gamma n_t \alpha (n_t)$ is assumed to be strictly increasing and concave in terms of $n_t$. For our purpose let $\alpha (n_t) = n_t^{-\epsilon}$ with $0 < \epsilon < 1$, that is the degree of altruism is of constant elasticity with respect to the number of children. At time $t = 0$, assume that there are $N_0$ identical individuals thereafter called the patriarchs. Each of them gives birth to $n_0$ children. If $N_1$ is the population at $t = 1$, then $N_1 = N_0 n_0$. Similarly $N_2 = N_1 n_1 = N_0 n_0 n_1$, etc. And $N_t = N_0 \prod_{s=0}^{t-1} n_s$. Because the utility level of the agent’s children is itself a function of the utility level his own descendant, we can by substitution write the patriarchs’ utility function as

$$V_0 = \sum_{t=0}^{\infty} \delta^t N_t^{1-\epsilon} U(c_t),$$

where $N_t$ is the population at time $t$. The analogous of the patriarchs’ utility function in continuous time version (see Barro and Becker (1989)) is

$$V_0 = \int_0^{\infty} e^{-\rho t} N_t^{1-\epsilon} U(c_t) dt,$$

(1)

$0 < \rho < 1$, $\rho$ being the rate of time preference taken to be a constant parameter and $N_t = N_0 e^{\int_0^t (n_s-1) ds}$.  

\footnote{Here the rate of utility discount is composed of the rate of time preference adjusted by the family size and the degree of the patriarch altruism. In continuous time, if the utility discount factor is denoted by $\theta(N,t)$, then $\theta / \theta = -[\rho + \epsilon (n_t - 1)]$. The patriarchs' utility, $\int_0^\infty N_t \theta(N_t, t) U(c_t) dt$, can be written as $\int_0^\infty N_t e^{-\rho t} e^{-\epsilon \int_0^t (n_s-1) ds} U(c_t) dt$. Using the definition of $N_t$, this utility can be easily seen as (1). Note that $n_t$ is endogenous, therefore our formulation is reminiscent of Epstein (1986) which addressed a completely different question in dynamics.}
In view of focusing on the issue of endogenous population, we suppose a simple production process which requires essentially the use of labor—the supply of which depends on agent’s fertility decisions. The aggregate production in the economy can be written as \( F(L, N_t) \) where \( L \) represents a factor available in fixed quantity and \( N_t \), the population taken to be the labor in use. Since we shall often refer to Malthus it is natural to interpret \( L \) as the total amount of land of a given level of soil fertility.

At each time, production is allocated between consumption and the cost of having \( n_t \) children which includes parent time, education, food, etc. If we consider the marginal cost of having one children as a constant, we have the following resources constraint at time \( t \):

\[
F(L, N_t) = N_t c_t + \beta N_t m_t.
\]

In a deterministic framework, since \( \dot{N}_t = (n_t - 1) N_t \), equation (2) can be cast in terms of the following diffusion process:

\[
dN_t = \left[ \frac{1}{\beta} F(L, N_t) - \frac{1}{\beta} N_t c_t - N_t \right] dt, \quad N_0 \text{ given.}
\]

In a stochastic framework this process is assumed to take the form of a Itô’s stochastic differential equation

\[
dN_t = \left[ \frac{1}{\beta} F(L, N_t) - \frac{1}{\beta} N_t c_t - N_t \right] dt + (\xi N_t)^{1/2} dZ, \quad N_0 \text{ given.}
\]

The problem of individual living at time \( t \) is to chose his consumption \( c_t \) and his number of offspring \( n_t \) under the budget constraint (see footnote 2). Since his utility depends not only on his own consumption but also on the well being of all of his descendants this recurrent relationship allows us to subsume utilities of

\footnote{This is a short-hand representation of the market allocation: on the factors supply side, assume that each individual is entitled to have an equal right on land, i.e. \( \frac{N_t}{X_t} \), and owns one unit of labor. On factors demand side, assume that the production in the economy is confered to a profit maximizing firm which pays labor and rents land at the marginal productivities \( w = F_N \) and \( r = F_L \). Factors market clearing conditions and a production under constant returns to scale imply \( y = w + r \frac{F(1)}{N_t} = \frac{F(1)}{N_t} \), where \( y \) denotes the individual income. The budget constraint for each individual is therefore \( \frac{F(1)}{N_t} = c_t + \beta n_t \) which amounts to nothing but the aggregate constraint (2) in the text.}

\footnote{For a complete analysis in this stochastic framework, see Section 4.}
immediate as well as all ensuing descendants in a single patriarch’s utility function (1). Note that this function satisfies the Strotz-consistency requirement hence all individual decisions could be studied by examining the patriarch’s decisions about the number of offspring and of consumption stream over the whole time horizon. This problem therefore consists of

$$\max_{\{c_t\}_{t=0}^{\infty}} \int_0^{\infty} e^{-\rho t} N_t^{1-\epsilon} U(c_t) \, dt$$  \hspace{1cm} (P)

subject to (3) or alternatively to (4),

which is a planning problem. It may be worthwhile to put forth the following remarks:

1- ”People” as the sole form of capital being only conceivable in pre-industrialist society constitutes the unique vehicle of saving. In continuous time each individual lives one (however small) lap of time thus this saving which should do more than offsetting the (instantaneous) depreciation of capital in order to avoid a complete vanishing of the society. With identical individuals this means $n_t > 0, t \in [0, \infty)$, must hold. Accordingly the constraint (2) can be rewritten as

$$c_t < F(L, N_t)/N_t.$$  \hspace{1cm} (5)

2- Problem (P) is a non-standard one-sector economic growth model. Here the population (state variable) is embedded in the objective function (1). Hence steady state equilibria if they exist may be multiple (see Kurz(1968)). In order to escape from the issue of multiplicity of equilibria and to highlight the problem of the Malthusian trap of subsistence consumption we adopt the following utility functional form:

$$U(c_t) = \begin{cases} (c_t - c_s)^{1-\sigma} / (1 - \sigma) & \text{if } c_t \geq c_s \\ -\infty & \text{if } c_t < c_s \end{cases}$$  \hspace{1cm} (A1)

where $c_s$ stands for the subsistence level below which the utility loss is infinite and $\sigma$, a constant stands for the elasticity of substitution. This form is known in the literature as the Stone-Geary utility function. Of course our problem should take into account the additional constraint $c_t \geq c_s$. We now turn to the analysis of our problem under the deterministic setting.
3. Endogenous Population under Deterministic Growth

Although not quite essential, we now assume that the production is of Cobb-Douglas type to ease calculations. Thus

\[ F(L, N_t) = L^{1-\gamma} N_t^\gamma = AN_t^\gamma, \quad 0 < \gamma < 1, \quad (A2) \]

so that our problem is

\[
\begin{align*}
\max_{\{c_t\}} \int_0^\infty e^{-\rho t} N_t^{1-\epsilon} \left[ (c_t - c_s)^{1-\sigma}/(1-\sigma) \right] dt \\
\text{subject to} \quad dN_t = \left[ \frac{1}{\beta} AN_t^\gamma - \frac{1}{\beta} N_t c_t - N_t \right] dt, N_0 \text{ given} \\
c_t - c_s \geq 0; n_t > 0.
\end{align*}
\]

If \( \{c_t, N_t\}_{t=0}^\infty \) is the optimal path, then it satisfies Pontryagin conditions:

\[
\begin{align*}
N_t^{1-\epsilon} (c_t - c_s)^{-\sigma} + \lambda - \frac{\pi_t N_t}{\beta} &= 0, \quad (6) \\
\bar{\pi}_t &= \rho \pi_t - \frac{(1-\epsilon)}{(1-\sigma)} N_t^{-\epsilon} (c_t - c_s)^{1-\sigma} - \pi_t \left[ \frac{\gamma}{\beta} AN_t^{\gamma-1} - \frac{1}{\beta} c_t - 1 \right], \quad (7) \\
N_t &= \frac{1}{\beta} AN_t^\gamma - \frac{1}{\beta} N_t c_t - N_t, \quad (8) \\
\lambda_t [c_t - c_s] &= 0; \quad \lambda_t \geq 0; \quad c_t - c_s \geq 0, \quad (9) \\
\lim_{t \to \infty} e^{-\rho t} \pi_t N_t &= 0; \quad \lim_{t \to \infty} e^{-\rho t} \pi_t \geq 0; \quad \lim_{t \to \infty} N_t \geq 0. \quad (10)
\end{align*}
\]

where \( \pi_t \) represents the efficient price of capital (one unit of population or equivalently a child). Equation (6) is an arbitrage condition between consumption and saving. The latter being made by the individual through having children. It stipulates that at each point in time the current marginal cost in terms of consumption must be equal to the current marginal value of a child. On the other hand, equation (7) appropriately rewritten stipulates that the net total rate of return on the investment in offspring must be equal to the rate of time preference. Note from A2 that the marginal utility of consumption is infinite at the subsistence level \( c_s \), hence we must have an interior solution i.e. \( c_t - c_s > 0 \), and therefore the Lagrangian multiplier \( \lambda_t = 0 \) for all \( t \). Finally, recalling remark 1 in the preceding section, \( \Gamma n_t > 0 \), and thus (5) should delimit the feasible region in the plane \( (c_t, N_t) \) for the optimal path.
In order to characterize in further details this optimal path in the phase plane $(c_t, N_t)$ let us derive $\dot{c}_t / c_t$ from (6) and (7). Differentiate totally (6) with respect to $t$ then substitute into (7) we get

$$
-\sigma \dot{c}_t / (c_t - c_s) = (1 + \rho - \epsilon) - \frac{1 - \epsilon}{\beta (1 - \sigma)} (c_t - c_s) + \frac{1 - \epsilon}{\beta} c_t + \frac{\epsilon - \gamma}{\beta} AN_t^{\gamma - 1}.
$$

Note that $\dot{c}_t = \dot{N}_t = 0$ would readily imply $\dot{c}_t = 0$, thus the steady state equilibrium could be determined by the locus $\Phi(c, N) \mid_{\dot{N}_t=0}$ and $\Psi(c, N) \mid_{c_t=0}$. The locus $\Phi$ is given by

$$
c_t + \beta - AN_t^{\gamma - 1} = 0,
$$

while the locus $\Psi$ is given by

$$
c_t - \frac{\beta (1 + \rho - \epsilon)}{1 - \epsilon} \frac{1 - \sigma}{\sigma} \frac{c_s - \epsilon - \gamma}{1 - \epsilon} \frac{1 - \sigma}{\sigma} AN_t^{\gamma - 1} = 0.
$$

Solve these two equations to get the steady state equilibrium values

$$
c_\infty = \frac{1}{\sigma (1 - \epsilon) + (1 - \sigma) (\gamma - \epsilon)} \left[ \beta (1 - \sigma) (1 + \rho - \gamma) + (1 - \epsilon) c_s \right],
$$

$$
N_\infty = \bar{L} \left[ \frac{1}{\sigma (1 - \epsilon) + (1 - \sigma) (\gamma - \epsilon)} \left[ \beta (1 - \sigma) (1 + \rho - \gamma) + (1 - \epsilon) c_s + \beta \right] \right]^{\frac{1}{\gamma - 1}}.
$$

At this point it is useful to state

**Lemma 3.1.** The interior solution path $\{c_t, N_t\}_{t=0}^\infty$ with $c_t > c_s$ satisfying (6)-(10) is optimal and converges monotonically to the steady state equilibrium which is unique and exhibits the saddle point property.

**Proof.** Note that the argument in the objective function corresponding to $c_t > c_s$ is strictly concave in $(c, N)$. And so is the state equation (8) therefore for sufficiency Theorem 9.3.1 and for convergence Theorem 9.5.1 in Leonard and Long (1992 p 288-295) applies. The uniqueness of the steady state trivially follows from its determination in the above.

**Lemma 3.2.** The optimal path does not admit $\sigma \geq 1$. 

7
Along the optimal path $\Gamma$ the steady state per capita consumption will be strictly greater than the subsistence level $c_s > 0$. Using (13) this implies $\beta(1 + \rho - \gamma)(1 - \sigma) > c_s(1 - \gamma)(\sigma - 1)$ which is impossible with $\sigma \geq 1$.

We now proceed to use the familiar phase diagram to characterize the optimal path and its dynamic properties. At the outset recalling that in positive orthant of the plane $(c, N)$ all feasible paths are contained the region delimited by (5) and $c_t - c_s \geq 0$. As for the locus $\Psi$, one can easily observe that it has a positive slope when $\epsilon < \gamma$ and a negative slope when $\epsilon > \gamma$. Since the dynamics involved in this analysis is not essentially different in the two cases we concentrate on the former which has a positive slope. On the other hand the locus $\Phi$ has a negative slope. Not surprisingly the steady state equilibrium clearly exits and is unique. Figure 3.1 shows the phase diagram analysis of the problem.

At this point it is interesting to see how the degree of altruism $\epsilon$ does affect the steady state equilibrium. Under the specification given earlier $\epsilon = 0$ means that
the parent attaches a weight equal to unity for their children’s utility regardless of the size of the family. The objective function (1) of the patriarch is Benthamite (sum-of-utilities). As \( \epsilon \) increases, this weight declines and the parent is said to be less altruistic. When \( \epsilon = 1 \), only the average utility of each child counts for the parent’s and the objective function (1) of the patriarch is Millian (see NerloveRazin and Sadka (1986)). Consider now a decrease in the value of \( \epsilon \) within the open interval \((0, 1)\) which amounts to an increase in the degree of altruism. Using again the phase diagram pictured in Figure 3.1, the locus \( \Phi \) given by (11) remains unchanged. However, the locus \( \Psi \) shifts down and to the right as one can obtain from (12) \( \partial \alpha_t/\partial \epsilon > 0 \) and \( \partial N_t/\partial \epsilon < 0 \). Thus the per capita consumption would be lower, and the population level in the long run higher, with an increase in altruism (and conversely). This is the so-called Edgeworth conjecture which clearly withstands in our model.  

Because of the constraint (5) depicted as the accessible frontier \( \Gamma(c, N) \) which delimits the region in which \( c_t - AN_t^{\gamma - 1} < 0 \), the dynamics of the system exhibits some peculiar but interesting characteristics. Although the steady state equilibrium is stable in the sense of a saddle point, the convergence toward it is not independent of the initial condition as with the standard one-sector growth model. Whenever this convergence to the optimal steady state is impeded, another outcome, namely the Malthusian equilibrium, is shown to prevail.  

Let \( \phi(c, N) = 0 \) denote the trajectory \( \{c_t, N_t\} \) which describes the saddle-path converging to the steady state \( E \). This saddle path clearly have a positive slope. Since \( \Gamma(c, N) \) has a negative slope, their intersection is uniquely determined at \( \widetilde{N} \).  

\[ A\widetilde{N}^{-1} = 0. \]  

If \( N_0 < \widetilde{N} \), one can always chose the optimal initial consumption located on the saddle path which insures the convergence of the economy to the optimal steady state equilibrium \( E \).  

This dynamics would obviously be precluded if \( N_0 > \widetilde{N} \). The system in this case would reach the subsistence level \( c_s \) at some time thereafter, \( \lambda_t \) becomes positive, and the population \( N_t \) would move to the Malthusian equilibrium outcome \( N_M \). It is not at all difficult to see that this Malthusian outcome \( M \) is stable, acting therefore as an attractor when the initial condition of our problem would not permit the convergence to the optimum \( E \). To sum up  

**Proposition 3.3.** Under the assumptions A1 and A2, the endogenous population is always bounded. If \( N_0 < \widetilde{N} \), the population dynamics follows the saddle path  

\[ c_t - AN_t^{\gamma - 1} < 0. \]  

\(^{4}\text{In the context of endogenous growth, Razin and Yuen (1995) provide a quite interesting tradeoff between population growth and income growth. Instead of the level, it is the rate of population growth which is higher under Benthamite utility than under Millian utility.}\)
which converges monotonically to the optimal steady state equilibrium where the per capita consumption exceeds the subsistence level. If, however, \( N_0 > \tilde{N} \), the population tends to the Malthusian equilibrium where a higher population level would have to bear the subsistence consumption.

It may be now worth to relate our analysis to Malthus’s vision. Since the publication of his "Essays in the Principle of Population" a little less than two centuries ago different interpretations of this vision are still at the heart of the debatable issue of economic sustainability. In what follows we shall call Malthusian when the population size despite possible technological progress as time unfolds comes indeed ultimately to equilibrate with the subsistence level of consumption. This outcome being clearly persistent over time is given the name of the Malthusian trap. According to the above Proposition only when the initial population size is large enough that the economy would be locked in Malthusian equilibrium. Still the question of Malthusian trap remains open and deserves a further examination.

Let us now consider a technological shift which like manna from heaven occurs once and for all at some time. The simplest way to take this into account is assigning a parameter \( \tau \) to our production function \( \tau AN_\tau \), where \( \tau \in [1, \bar{\tau}] \). The technological shift will exert a horizontal translation of the loci \( \Phi, \Psi \) as well as \( \Gamma \) to the right (see Figure 3.2). Recalling the steady state equilibrium values \( c_\infty \) and \( N_\infty \) given by equations (13) and (14) for which \( \tau = 1 \). It is easy to check that \( c_\infty (\tau) = c_\infty \), and \( N_\infty (\tau) = \tau^{1/1-\gamma} N_\infty \). Suppose that the economy is locked in the Malthusian equilibrium \( N_M = \bar{L} N_\gamma^{-1} \). The question we now ask is whether the economy can escape from the Malthusian trap once a technological shift occurred. For this purpose let us define parametrically the function \( \tilde{N}(\tau) \) by the equation \( \phi (\tau AN_\gamma^{-1}, \tilde{N}) = 0 \). We can state

**Proposition 3.4.** If there exists a technological shift parameter \( \tau^* \in [1, \bar{\tau}] \) such that \( \tilde{N}(\tau^*) > N_M \), then the economy would escape from the Malthusian equilibrium and tends toward the new steady state defined by \( c_\infty (\tau^*) \) and \( N_\infty (\tau^*) = \tau^{(1/1-\gamma)} N_\infty \). Otherwise, the Malthusian equilibrium is persistent.

**Proof.** First note that \( d\tilde{N}(\tau)/d\tau > 0 \) because of the horizontal translation induced by the technological shift. Moreover the saddle path also shifts to the right as \( \phi^S (c, N) = 0 \). Now simply use Proposition 3.3 to obtain the convergence of the economy to the steady state \( E^* \) and this completes the proof. ■
This Proposition is somewhat evident in itself merits two comments. First, it demonstrates that Malthus’s pessimism is not quite justified when the family size is part of decisions made by rational individual agents. Even captured by the Malthusian trap for some time there is a chance for the economy to get out of the misery provided that sufficient technological progress could be realized. Second, it might be of interest to make this progress itself endogenous in the present model. Not as manna that falls from some rather obscure heaven the technological advance is but the fruit of R&D efforts which are by no mean free to the society.

In the meantime, assume by now that there is no technological progress and $N_0 > \tilde{N}$ so that the Malthusian trap is unavoidable. Are there any policies which might help the economy escaping from the miserable subsistence level of

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{Figure32.png}
\caption{Technological Shift}
\end{figure}
consumption? The answer to the above question naturally comes from public interventions through appropriately modifying the cost of child-rearing by a tax (subsidy) policy. Let $\theta$ denote this policy instrument with $\theta > 0$ if it is a subsidy per child and conversely. The net cost of child rearing is therefore $\beta^\theta = \beta - \theta$. All ingredients in this analysis can now be parametrized by $\theta$. Clearly both loci $\Psi$ and $\Phi$ shift to the right as $\theta$ increases but the accessible frontier $\Gamma$ remains unchanged. Let $\overline{\theta} = \beta - \varepsilon$, $\varepsilon$ being positive and however small, and define $\overline{N}(\overline{\theta})$ as the intersection of $\Psi(\overline{\theta})$ and $\Gamma$ (see Figure 3.3). Since $\Phi(\overline{\theta})$ lies in the vicinity of $\Gamma$ in the plane $(c, N)$, it follows that $N_{\infty}(\overline{\theta}) = \overline{N}(\overline{\theta})$. We are ready to show

**Proposition 3.5.** Provided that the initial population $N_0 \leq \overline{N}(\overline{\theta})$, it is always possible to escape from the Malthusian trap through the subsidy of child rearing.

**Proof.** Since the locus $\Psi$ shifts rightward as $\theta$ increases it is immediate that $\overline{N}(\theta)$ is increasing in $\theta$. Hence there exists $\overline{\theta} \leq \overline{\theta}$ which satisfies $N_0 = \overline{N}(\overline{\theta})$. It suffices to chose now a subsidy $\theta$ in the right neighborhood of $\overline{\theta}$ and apply Proposition 3.3.

From (13) and (14) a little handling yields $dc_\infty/d\theta < 0$ and $dN_{\infty}(\theta)/d\theta > 0$. In Figure 3.3 the locus $\Psi(\theta)$ shifts more considerably than does $\Phi(\theta)$ as a result of implementing $\theta$. Also the saddle path uniformly shifts to the right as $\phi^\theta(c, N) = 0$ so that the economy would converge to the steady state $E^\theta$. Child-rearing cost subsidy could be financed by a per capita lump sum tax which is neutral. This cost subsidy would discourage consumption but bring about a new level of steady state equilibrium level of population reachable from an initial condition which otherwise would have led to a Malthusian outcome. Although apparently paradoxical in static sense because one requires the cost subsidy of child-rearing in a situation qualifying as over-populated this result can be understood in dynamics as with many policy problems in capital theory (say throwing away some capital good in view of reaching the Golden Rule equilibrium). We have not yet selected the best cost subsidy policy leaving this matter for another occasion.

4. **Endogenous Population under Stochastic Growth**

According to a wisdom which is repeatedly recalled in Buddhism”Life is uncertain but Death sure”. A child in our model is now faced with some probability of premature death assumed to be $1 - p$, where for simplicity $0 < p < 1$ is a
Figure 3.3: Child-Rearing Subsidy
constant and represents the probability of survival. Let \( E \) denotes the expectation operator, then 
\[
E[dN_t] = (pn_t - 1) N_t dt = \left[ \frac{1}{\beta} AN_t^\gamma - \frac{1}{\beta} N_t c_t - N_t \right] dt
\]
where \( \bar{\beta} = \beta / p \) corresponds to the expected cost of a surviving child. Assume by now that the population growth undergoes a white-noise stochastic process with a mean \( E[dN_t] \) and a variance which is proportional to the number of living persons \( \text{Var}[dN_t] = \xi N_t dt \). This process can be conveniently described by the familiar Ito’s stochastic differential equation:
\[
dN_t = \left[ \frac{1}{\beta} AN_t^\gamma - \frac{1}{\beta} N_t c_t - N_t \right] dt + (\xi N_t)^{1/2} dZ,
\]  
(4)
where \( dZ \sim N(0, dt) \) is a Brownian motion.

We have stated our problem (P) under stochastic population growth given by (4) in Section 2. This problem will now be solved. For this purpose we shall use the analytical apparatus provided by Bourguignon (1974) and Merton (1975, 1990). We first derive the probability density function of the population at the stochastic steady state and then use it to find the optimal stochastic consumption function.

For the stochastic process (4) let the transition probability be
\[
G(N, t) = \text{Pr}[N_t \leq N | N_0],
\]
where \( N \) is random. We make the assumption that \( G(N, t, N_0) \) have a conditional probability density \( g(N, t, N_0) \) which is well defined on the interval \([0, \infty)\).

Following closely Merton (1990, pp. 601-604) the variation in the probability density during \( dt \) is described by the following Kolmogorov-Fokker-Planck forward equation:
\[
\frac{1}{2} \frac{\partial^2}{\partial N^2} \left[ \xi N_t g(N, t, N_0) \right] - \frac{\partial}{\partial N} \left[ \left( \frac{1}{\beta} AN_t^\gamma - \frac{1}{\beta} N_t c_t - N_t \right) g(N, t, N_0) \right] = \frac{\partial g(N, t, N_0)}{\partial t}.
\]  
(15)
It is theoretically possible to obtain a full description of the process over the time horizon by integrating equation (18) subject to the following initial conditions:
\[
g(N, 0, N_0) = \begin{cases} 
1, & N = N_0 \\
0, & N \neq N_0
\end{cases}
\]  
(16)
On the other hand if the stochastic steady state exists \( g(N, t, N_0) \) becomes time independent. Let us denote the probability density function at the stochastic
steady state by \( g(N) \). We can then find the steady-state density \( g(N) \) by solving:
\[
\frac{1}{2} \frac{d^2}{dN^2} [\xi Ng(N)] - \frac{d}{dN} \left[ \left( \frac{1}{\beta} AN^\gamma - \frac{1}{\beta} Nc(N) - N \right) g(N) \right] = 0. \tag{17}
\]
The solution of this differential equation is
\[
g(N) = \frac{m_1}{\xi N} \exp \left\{ 2 \int_0^N \frac{Ay^\gamma - yc(y) - \beta y}{\beta y} dy \right\} + \frac{m_2}{\xi N} \int_0^N \exp \left\{ 2 \int_y^N \frac{As^\gamma - sc(s) - \beta s}{\beta s} ds \right\} dy.
\tag{18}
\]

Mandl (1968)\(^5\) states a theorem on the existence of the stochastic steady state. This theorem assumes that the bounds of the distribution are inaccessible. This mean that \( \lim_{x \to 0} \Pr [N_t \leq x] = 0 \) and \( \lim_{x \to \infty} \Pr [N_t \geq x] = 0 \). Intuitively if \( \lim_{x \to \infty} \Pr [N_t \geq x] \neq 0 \) there will be no stochastic steady state because \( N = \infty \) becomes an absorbing bound. On the other hand if \( \lim_{x \to 0} \Pr [N_t \leq x] \neq 0 \) there will exist a \( t \) such as \( N_t = 0 \), \( \forall t \geq t \) a degenerate case not considered in this paper\(^6\). These bounds are inaccessible if and only if:

\[
\begin{align*}
\int_0^N \frac{1}{\xi N} \int_x^N \exp \left\{ 2 \int_y^N \frac{Ay^\gamma - yc(y) - \beta y}{\beta y} dy \right\} dy dx &= \infty \Gamma \\
\int_N^\infty \frac{1}{\xi N} \int_x^N \exp \left\{ 2 \int_y^N \frac{Ay^\gamma - yc(y) - \beta y}{\beta y} dy \right\} dy dx &= \infty , \\
\int_0^\infty \frac{1}{\xi N} \exp \left\{ 2 \int_x^N \frac{Ay^\gamma - yc(y) - \beta y}{\beta y} dy \right\} dx &= \infty .
\end{align*}
\tag{A3}
\]

These three conditions stipulated as A3 imply that \( m_2 = 0 \) and \( m_1 \neq 0 \). We now obtain

**Lemma 4.1.** Under A3, the population at the stochastic steady state has the following probability density function
\[
g(N) = \frac{m}{\xi N} \exp \left\{ 2 \int_0^N \frac{Ay^\gamma - yc(N) - \beta y}{\beta y} dy \right\}, \tag{19}
\]
where \( m \) is such that \( \int_0^\infty g(N) dN = 1. \)


\(^6\)It is possible to have \( N = 0 \) as an accessible bound and under some conditions a stochastic steady state still exists. Bourguignon (1974) gives a discussion on those conditions.
Given the probability density function of the population at the stochastic steady state, we can find an asymptotic approximation of the consumption function. For a problem in which the utility function is time independent, Merton (1975) has shown that solving the stochastic problem over the infinite time horizon is equivalent to maximizing $E \left[ N^{1-\epsilon} \frac{(c_c - c_s)^{1-\sigma}}{1-\sigma} \right]$ \footnote{See also Merton (1990, p 392-398) on a lucid account of the stochastic Ramsey problem.}. By solving this latter we obtain $\hat{c}(N)$ which is the optimal consumption rule at the stochastic steady state for a time independent utility function.

In the time independent utility framework, the patriarch faces the following problem:

$$
\max_{\{c_N\}_{N=0}^{\infty}} \int_0^{\infty} N^{1-\epsilon} \frac{(c_c - c_s)^{1-\sigma}}{1-\sigma} g(N) \, dN \tag{20}
$$

subject to

$$
\int_0^{\infty} g(N) \, dN = 1.
$$

The solution to this problem is reported in the Appendix A.1 where we obtain an equation which describes $\hat{c}(N)$ implicitly:

$$
0 = (\hat{c}(N) - c_s)^{-\gamma} N^{-\epsilon} \left[ -\frac{\epsilon}{N} + \frac{2}{\xi \beta} \left[ AN^{\gamma-1} - \hat{c}(N) - \bar{\beta} \right] \right]
$$

$$
+ \frac{2}{N \xi \beta} \left\{ N^{1-\epsilon} \frac{(\hat{c}(N) - c_s)^{1-\sigma}}{1-\sigma} - E \left[ N^{1-\epsilon} \frac{(\hat{c}(N) - c_s)^{1-\sigma}}{1-\sigma} \right] \right\}.
$$

It is impossible to solve (21) for $\hat{c}(N)$. However, we can still get some useful information by closely examining this equation at the certainty equivalent population. We now state

**Proposition 4.2.** When the population growth follows a Brownian diffusion process, the saving rate in terms of offsprings number would increase by a component identified as the precautionary saving.

**Proof.** Let us define the certainty equivalent population level as $\tilde{N}$ which satisfies $\tilde{N}^{1-\epsilon} \frac{(c_c - c_s)^{1-\sigma}}{1-\sigma} = E \left[ \tilde{N}^{1-\epsilon} \frac{(c_c - c_s)^{1-\sigma}}{1-\sigma} \right]$. From (21) we readily get the certainty equivalent consumption equation

$$
\hat{c}(\tilde{N}) = AN^{\gamma-1} - \bar{\beta} - \frac{\epsilon \xi \bar{\beta}}{2\tilde{N}}.
$$
where the consumption is reduced by the last term of the right hand side. Thus the certainty equivalent rate of saving should be increased by the same amount.

We note that \( \lim_{\xi \to 0} \hat{c}(\hat{N}) = A\hat{N}^{\gamma - 1} - \bar{\beta} \). This means that when the variance of the stochastic process vanishes and the probability of survival equals 1, we have exactly the same consumption rule obtained for the deterministic steady state. One reasonably expect that \( \int_0^\infty Ng(N)dN \geq N_{\infty} \) and the optimal stochastic consumption \( c(N) \) as well as the optimal number of offsprings \( n(N) \) vary with the risk that results from the stochastic nature of (21). Unfortunately, we have not been able to provide a throughout analysis of these points. But we can use numerical method to approximate the optimal consumption rule around the deterministic steady state. To do this, we follow a method suggested by Judd and Guu (1993) which allows us to obtain the approximated optimum consumption of our model. In Appendix A.2, using the following values for different parameters: \( \sigma = 0.9, \epsilon = 0.7, \rho = 0.05, \gamma = 0.8, \beta = 1 \) and \( L = 0.005 \), we get

\[
c(N) = 0.0892857 + 1.6515 (N - 0.00326033)^2 + 17664.8 (N - 0.00326033)^3 - 5.99113 \times 10^6 (N - 0.00326033)^4
\]

under deterministic conditions and

\[
c(N, \xi) = 0.0892857 + 1.6515 (N - 0.00326033)^2 + 92.072 (N - 0.00326033)^3 + 556.012 \xi + 230655 (N - 0.00326033) \xi - 8.18043 \times 10^6 \xi^2
\]

under stochastic conditions. Using the latter equation we can draw Figure 4.1 and Figure 4.2 which show the behavior of the consumption and the number of offsprings as a function of the variance of the stochastic process in the neighborhood of the deterministic steady state. We can then observe that an increase of this variance induces the agent to reduce consumption and to increase the number of their children. This property is quite robust with respect to different chosen values of the parameters above mentioned.

5. Concluding Comments

In this paper, we have shown that when fertility decisions are endogenous in a model where human beings constitute the sole capital asset the economy would run into the Malthusian equilibrium with high population and subsistent per capita consumption only when the initial population is large enough. Otherwise, the
Figure 4.1: Consumption
Figure 4.2: Number of offsprings
steady state equilibrium with lower population and higher per capita consumption than the Malthusian’s outcome would prevail. Also we have qualified the persistent Malthusian equilibrium - or Malthusian trap - and demonstrated that the economy can escape from this unappealing long run situation either through a suitable technological shift or an appropriate subsidy of the child-rearing cost. Our results are then extended to the stochastic framework where the time rate of population change undergoes a process of Brownian motion. The probability distribution of the steady state is well determined, and saving via the number of offsprings as with the standard capital model incorporates a precautionary component.

For the purpose of exposition and of simplifying computations we adopt the Cobb-Douglas specification of the production function but this assumption is only crucial for the sub-section 4.2 where we need to find the consumption (thus saving) function at the stochastic steady state for the time independent utility case. As for the rest all results withstand with a more general specification. The assumption of a Stone-Geary utility function is on the contrary more restrictive. With a general utility function we expect to get multiple steady state equilibria in our model and consequently the study of the dynamics there involved should become much more demanding. Recent advances in the study of complex non-linear dynamics would suggest the possibility of cyclical behavior and even of chaos in such model. This issue seems to deserve a separate investigation.

Among others generalizations that we would like to mention we first refer to the endogeneization of the technological progress. One attempt is given in Amigues and Hung (1996) where one fraction of time that disposes each individual agent could be devoted to ”education” of their children which then enhances the accumulation of human capital. In this case population would grow without any upper bound a conclusion which is excluded in the present paper. There are of course many others avenues as well. Another aspect which seems to us quite interesting is to treat the carrying capacity not as a fixed quantity of land as with the present model but rather as an environmental stock which is depleted with its use. This would set up the link population-environment which to our knowledge is still the missing link in the recent literature on economic sustainability.
Appendix

A.1 Computing the time independent stochastic consumption function at the stochastic steady state.

In the time independent utility framework the patriarch faces the following problem:

$$\max_{\{c\}_N} \int_0^\infty N^{1-\epsilon} \frac{(c_N - c_s)^{1-\sigma}}{1 - \sigma} g(N) \, dN$$

subject to

$$\int_0^\infty g(N) \, dN = 1.$$ 

Defining $b(N) = \int^N \frac{\delta y - y \gamma(y)}{y} dy$ we have $g(N) = \frac{m}{\xi N} \exp \left\{ \frac{2}{\beta \xi} b(N) \right\}$. Moreover note that $b'(N) = AN^{-1} - c(N) - \bar{\beta}$ and $b''(N) = (\gamma - 1)AN^{-2}$. We can then rewrite the problem as:

$$\max \int_0^\infty N^{1-\epsilon} \left[ AN^{-1} - b'(N) - \bar{\beta} - c_s \right]^{1-\sigma} \frac{m}{\xi N} \exp \left\{ \frac{2}{\beta \xi} b(N) \right\} \, dN$$

$$- \lambda \left[ 1 - \int_0^\infty \frac{m}{\xi N} \exp \left\{ \frac{2}{\beta \xi} b(N) \right\} \, dN \right].$$

Euler’s equations of this problem are:

$$0 = - \frac{d}{dN} \left[ N^{-\epsilon} \left[ AN^{-1} - b'(N) - \bar{\beta} - c_s \right]^{1-\sigma} \frac{1}{\xi} \exp \left\{ \frac{2}{\xi \beta} b(N) \right\} \right]$$

$$+ \frac{2}{\xi^2 \beta N} \exp \left\{ \frac{2}{\xi \beta} b(N) \right\} \left[ N^{1-\epsilon} \left[ AN^{-1} - b'(N) - \bar{\beta} - c_s \right]^{1-\sigma} \frac{1}{1 - \sigma} - \lambda \right], \quad (A.1)$$

$$0 = \int_0^\infty N^{1-\epsilon} \frac{(c - c_s)^{1-\sigma}}{1 - \sigma} g(N) \, dN - \lambda \int_0^\infty g(N) \, dN, \quad (A.2)$$

$$0 = 1 - \int_0^\infty \frac{m}{\xi N} \exp \left\{ \frac{2}{\xi \beta} b(N) \right\} \, dN. \quad (A.3)$$
From (A.2) and (A.3) we can deduce \( \lambda = E \left[ N^{1-\epsilon} \frac{(\hat{c}(N) - c_s)^{1-\sigma}}{1-\sigma} \right] \). Computing the derivative in equation (A.1) and substituting for \( b' (N) \Gamma b'' (N) \) and \( \lambda \Gamma \) we obtain:

\[
0 = (\hat{c}(N) - c_s)^{-\sigma} N^{-\epsilon} \left[ -\frac{\epsilon}{N} + \frac{2}{\xi \beta} \left[ AN^{\gamma - 1} - \hat{c}(N) - \hat{\beta} \right] \right]
+ \frac{2}{\xi \beta N} \left\{ N^{1-\epsilon} \frac{(\hat{c}(N) - c_s)^{1-\sigma}}{1-\sigma} - E \left[ N^{1-\epsilon} \frac{(\hat{c}(N) - c_s)^{1-\sigma}}{1-\sigma} \right] \right\}.
\]

**A.2 Computing the numerical approximation of the stochastic consumption function**

We want to compute the following Taylor series approximation of the stochastic consumption function \( \Gamma c(N, \xi) \Gamma \) around the deterministic steady state \((N_\infty, 0)\):

\[
c(N, \xi) = c(N_\infty, 0) + c_N(N_\infty, 0) (N - N_\infty) + c_\xi(N_\infty, 0) \xi + \frac{1}{2} c_{NN}(N_\infty, 0) (N - N_\infty)^2 + \frac{1}{2} c_{\xi\xi}(N_\infty, 0) \xi^2 + c_{N\xi}(N_\infty, 0) \xi (N - N_\infty). \tag{A.4}
\]

What is interesting with the Taylor series approximation is that we don’t have to know the exact form of \( c(N, \xi) \) but only the numerical values of its derivatives at \((N_\infty, 0)\). To find these values we follow closely the method suggested by Judd and Guu (1993).

In a first step we need to find the numerical values of \( c_N(N_\infty, 0) \Gamma c_{NN}(N_\infty, 0) \Gamma c_{N\xi}(N_\infty, 0) \Gamma c_{N\xi\xi}(N_\infty, 0) \Gamma \). In order to evaluate those derivatives we need to use the following Taylor series approximation of the deterministic problem (Q):

\[
c(N) = c(N_\infty) + c'(N_\infty) (N - N_\infty) + \frac{c''(N_\infty)}{2} (N - N_\infty)^2 + \frac{c'''(N_\infty)}{6} (N - N_\infty)^3 + \frac{c^{(4)}(N_\infty)}{24} (N - N_\infty)^4. \tag{A.5}
\]

The Bellman equation for our dynamic programming problem is

\[
0 = \max_c \lambda N^{1-\epsilon} \frac{c^{1-\sigma}}{1-\sigma} - \rho V(N) + V'(N) \left[ \frac{1}{\beta} AN^{\gamma} - \frac{1}{\beta} \Gamma N \right]. \tag{A.6}
\]

---

*We need all those values to compute \( c_{N\xi} \) and \( c_{\xi\xi} \).*
The first-order condition for the indicated maximization is
\[ V'(N) = \beta N^{-\epsilon} c^{-\sigma}. \]  \[(A.7)\]

Equation \((A.7)\) implies that consumption can be expressed as a function of population \(c(N)\). Substituting \((A.7)\) into \((A.6)\), we get
\[ 0 = N^{1-\epsilon} c(N)^{1-\sigma} - \rho V(N) + V'(N) \left[ \frac{1}{\beta} AN^\gamma - \frac{1}{\beta} N c(N) - N \right]. \]  \[(A.8)\]

Equations \((A.7)\) and \((A.8)\) describe our problem. If we differentiate \((A.8)\) with respect to \(N\), we get
\[ 0 = (1 - \epsilon) N^{-\epsilon} c(N)^{1-\sigma} + V''(N) \left[ \frac{1}{\beta} AN^\gamma - \frac{1}{\beta} N c(N) - N \right] \]  \[(A.9)\]
\[ + V'(N) \left[ \frac{\gamma}{\beta} AN^\gamma - \frac{1}{\beta} c(N) - 1 - \rho \right]. \]

If we differentiate \((A.7)\) with respect to \(N\), we get
\[ V''(N) = -\epsilon \beta N^{-\epsilon} c(N)^{-\sigma} - \sigma \beta N^{-\epsilon} c(N)^{-\sigma - 1} c'(N), \]  \[(A.10)\]
which together with \((A.7)\) allows us to eliminate \(V'\) and \(V''\) and arrive at a single equation for \(c(N)\):
\[ 0 = \frac{\sigma (1 - \epsilon)}{1 - \sigma} c(N)^{-\sigma} c'(N) \left[ AN^\gamma - N c(N) - \beta N \right] + (\gamma - \epsilon) AN^{\gamma - 1} - \beta (1 + \rho - \epsilon). \]  \[(A.11)\]

We note that \((A.11)\) says that the Bellman equation is always zero. It is this fact which we will exploit since this implies that all the derivatives are also zero at all \(N\). We can then find the numerical values of \(c'(N)\) \(\Gamma c''(N)\) \(\Gamma c'''(N)\) and \(\Gamma c''(N)\) by successive derivation of \((A.11)\). If we use the following values for the problem's parameters: \(\sigma = 0.9\) \(\Gamma \epsilon = 0.7\) \(\Gamma \rho = 0.05\) \(\Gamma \gamma = 0.8\) \(\Gamma \beta = 1\) and \(\Gamma = 0.005\) \(\Gamma\) we then find the following Taylor series approximation of the deterministic consumption function
\[ c(N) = 0.0892857 + 1.6515 (N - 0.00326033) + 92.072 (N - 0.00326033)^2 \]  
\[ + 17664.8 (N - 0.00326033)^3 - 5.99113 \times 10^6 (N - 0.00326033)^4. \]
We can now proceed to the determination of the Taylor series approximation of the stochastic consumption function. The Bellman equation of the stochastic problem is

\[ 0 = \max_c N^{1-\epsilon} c^{1-\sigma} - \rho V(N) + V'(N) \left[ \frac{1}{\beta} AN^\gamma - \frac{1}{\beta} Nc - N \right] + V''(N) \xi N. \]  

(A.12)

The first-order condition remains given by equation (A.7). Substituting the optimal consumption \( c(N, \xi) \) into equation (A.12) we get

\[ 0 = N^{1-\epsilon} c(N, \xi)^{1-\sigma} - \rho V(N) + V'(N) \left[ \frac{\gamma}{\beta} AN^{\gamma-1} - \frac{1}{\beta} c(N, \xi) - 1 - \rho \right] + V''(N) \xi N. \]  

(A.13)

Equations (A.7) and (A.13) describe the stochastic problem. If we differentiate equation (A.13) with respect to \( N \) we get

\[ 0 = (1 - \epsilon) N^{-\epsilon} c(N, \xi)^{1-\sigma} + V'(N) \left[ \frac{\gamma}{\beta} AN^{\gamma-1} - \frac{1}{\beta} c(N, \xi) - 1 - \rho \right] + V''(N) \xi N. \]  

(A.14)

If we differentiate twice (A.7) with respect to \( N \) we get

\[ V''(N) = -\epsilon \beta N^{-\epsilon-1} c(N, \xi)^{-\sigma} - \sigma \beta N^{-\epsilon} c(N, \xi)^{-\sigma-1} c_N(N, \xi) \]  

(A.15)

and

\[ V'''(N) = \epsilon (1 + \epsilon) \beta N^{-\epsilon-2} c(N, \xi)^{-\sigma} + 2 \sigma \epsilon \beta N^{-\epsilon-1} c(N, \xi)^{-\sigma-1} c_N(N, \xi) \]  

(A.16)

\[ + \sigma (1 + \sigma) \beta N^{-\epsilon} c(N, \xi)^{-\sigma-2} c_N(N, \xi) - \sigma \beta N^{-\epsilon} c(N, \xi)^{-\sigma-1} c_{NN}(N, \xi), \]

which together with (A.7) allow us to eliminate \( V' \Gamma V'' \) and \( V''' \) and arrive at a single equation for \( c(N, \xi) \):

\[ 0 = \frac{\sigma (1 - \epsilon)}{1 - \sigma} c(N, \xi) + (\gamma - \epsilon) AN^{\gamma-1} - \beta \left( 1 + \rho - \epsilon + \epsilon \xi^2 \right) \]  

(A.17)

\[ - \sigma \frac{c_N(N, \xi)}{c(N, \xi)} \left[ AN^\gamma - Nc(N, \xi) - \beta N + \beta \xi \right] \]

\[ + \beta \xi \left[ \frac{\epsilon (1 + \epsilon)}{N} + 2 \sigma \frac{c_N(N, \xi)}{c(N, \xi)} + \sigma (1 + \sigma) \frac{c_N(N, \xi)}{c(N, \xi)^2} - \sigma \frac{c_{NN}(N, \xi)}{c(N, \xi)} \right]. \]
We note that like equation (A.11) equation (A.17) says that the Bellman equation is always zero. Once more we will exploit this fact since it implies that all the partial derivatives are also zero at all \( N \) and at all \( \xi \). We can then find the numerical values of \( c_\xi (N_\infty, 0) \) \( c_{\xi \xi} (N_\infty, 0) \) and \( c_{N \xi} (N_\infty, 0) \) by successive derivation of (A.17). If we use the same values for the parameters we find the following Taylor series approximation of the stochastic consumption function

\[
c (N, \xi) = 0.0892857 + 1.6515 (N - 0.00326033) + 92.072 (N - 0.00326033)^2
+ 556.012\xi + 230655 (N - 0.00326033)\xi - 8.18043 \times 10^6 \xi^2.
\]

Figure 4 is drawn by using this equation.
References


