Statistical Inference for the Measurement of the
Incidence of Taxes and Transfers

by

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Abstract

We establish the asymptotic sampling distribution of general functions of quantile-based estimators computed from samples that are not necessarily independent. The results provide the statistical framework within which to assess the progressivity of taxes and benefits, their horizontal inequity, and the change in the inequality of income which they cause. By the same token, these findings characterise the sampling distribution of a number of popular indices of progressivity, horizontal inequity, and redistribution. They can also be used to assess welfare and inequality changes using panel data, and to assess poverty when it depends on estimated population quantiles. We illustrate these results using micro data on the incidence of taxes and benefits in Canada.

On étudie les propriétés asymptotiques d’estimateurs calculés sur la base de quantiles d’un échantillon ou de plusieurs échantillons qui ne sont pas forcément indépendants, et on développe la distribution asymptotique d’une fonction arbitraire de tels estimateurs. Les résultats fournissent un cadre statistique dans lequel on peut évaluer la progressivité d’un système fiscal, en particulier son impact sur l’inégalité des revenus et l’inéquité horizontale qu’il peut induire. Ils permettent également de caractériser les distributions d’échantillonnage de la plupart des indices utilisés couramment pour mesurer la progressivité, la redistribution des revenus, l’inéquité horizontale et la pauvreté, ainsi que pour évaluer les changements de bien-être ou d’inégalité sur la base de données de panel. On se sert de données microéconomiques canadiennes sur l’incidence fiscale afin d’illustrer les résultats.

Keywords  Income inequality indices, Tax progressivity, Horizontal equity, Poverty indices, Distribution-free statistical inference.

Mots clés  Indices d’inégalité des revenus, Progressivité fiscale, Équité horizontale, Indices de pauvreté, Inférence statistique robuste.

JEL classification  C14, C40, D31, D63
1. Introduction

The last decade has seen considerable use and development of statistical theory for inferring the dominance of one distribution (of income, wealth, wages, etc.) over another. Various welfare criteria have been applied, such as first- and second-order stochastic dominance, Lorenz dominance, “transfer-sensitivity” dominance, and comparisons of “poverty” deficit curves (Beach and Davidson (1983), Bishop, Formby and Thistle (1992), Howes (1993), Beach, Davidson and Slotsve (1994)). All such criteria involve quite general principles of anonymity, efficiency, and equity (see, for instance, Shorrocks (1983) and Davies and Hoy (1994)). The comparisons typically seek to establish inequality and social welfare rankings using independently-drawn samples from the relevant populations.

We extend these developments to the measurement of redistribution, progressivity, and horizontal inequity. More generally, we establish the asymptotic sampling distribution of general functions of quantile-based estimators computed from samples that are not necessarily independent. Dependent samples may be due, for instance, to the correlation between gross and net income distributions, or to the correlation of incomes across time when panel data are used. The results thus provide the statistical framework within which to assess the progressivity of taxes and benefits, and the changes, in the inequality of income, or in the ranking of individuals with respect to income, which they may cause. Similarly, one can readily obtain the sampling distributions of a number of popular or recent measures of progressivity, horizontal inequity, and redistribution (see Musgrave and Thin (1948), Suits (1977), Reynolds and Smolensky (1976), Kakwani (1977), Atkinson (1979), Plotnick (1981), Pfähler (1987), Aronson et al. (1994), and Lerman and Yitzhaki (1995)). The results can also be applied to the impact on poverty indices of a tax and benefit system, or of other socio-economic phenomena, when such poverty indices depend on estimated population quantiles. They furthermore encompass as special cases most of the previous statistical inference results for the measurement of inequality and social welfare. Our results are distribution-free in the sense that they do not require a specification of the population distributions from which the samples are drawn.

For instance, the poverty line may be half of median income, or the weights on the incomes of the poor in the poverty index may depend on their ranking in the income distribution.
2. The Measurement of Progressivity, Horizontal Inequity, and Redistribution

Consider a population of households indexed by $\omega \in \Omega$. Let $X(\omega)$ denote the gross income of household $\omega$, $T(\omega)$ the tax burden of the household, and $M(\omega) = X(\omega) - T(\omega)$ its income net of taxes. We write the distribution on $\Omega$ as $F(\omega)$, so that mean gross income, $\mu_X$, is defined as

$$\mu_X = \int_{\Omega} X(\omega) \, dF(\omega).$$

The mean tax burden, $\mu_T$, and mean net income, $\mu_M$, are defined similarly. The Lorenz curve $L_X(p)$ for gross income is defined as

$$L_X(p) = \frac{1}{\mu_X} \int_{0}^{p} x \, dF_X(x), \quad \text{with} \quad p = F_X(y).$$

Here $F_X$ is the cumulative distribution function (c.d.f.) of gross income $X$:

$$F_X(x) = \Pr(X(\omega) \leq x) = \int_{\Omega} I_{\{X(\omega) \leq x\}}(\omega) \, dF(\omega),$$

where the indicator function $I_{\{X(\omega) \leq x\}}$ is defined by

$$I_{\{X(\omega) \leq x\}}(\omega) = \begin{cases} 1 & \text{if } X(\omega) \leq x; \\ 0 & \text{otherwise.} \end{cases}$$

Thus $L_X(p)$ can equivalently be defined as

$$L_X(p) = \frac{1}{\mu_X} \int_{\Omega} I_{\{X(\omega) \leq y\}}(\omega) \, x \, dF_X(x), \quad \text{with} \quad p = F_X(y).$$

The concentration curves $L_M$ and $L_T$ for net income and taxes are, respectively:

$$L_M(p) = \frac{1}{\mu_M} \int_{\Omega} I_{\{X(\omega) \leq y\}}(\omega) \, M(\omega) \, dF(\omega),$$

and

$$L_T(p) = \frac{1}{\mu_M} \int_{\Omega} I_{\{X(\omega) \leq y\}}(\omega) \, T(\omega) \, dF(\omega).$$

With $F^*$ the c.d.f. of $M$, the Lorenz curve for $M$ is analogously defined as:

$$L^*(p) = \frac{1}{\mu_M} \int_{0}^{p} x \, dF^*(x), \quad \text{with} \quad p = F^*(y)$$

$$= \frac{1}{\mu_M} \int_{\Omega} I_{\{M(\omega) \leq y\}}(\omega) \, M(\omega) \, dF(\omega).$$
$L_X(p), L_M(p), L_T(p)$ and $L^*(p)$ are the basis for much of the theory of the measurement of inequality, progressivity, horizontal inequity, and redistribution. For instance, it is well known that, if and only if $L^*(p)$ dominates $L_X(p)$, inequality under $X$ will be greater than under $M$ for all inequality measures that satisfy $P$: symmetry (or anonymity), mean independence, and the strict Dalton-Pigou principle of transfers.

Let $R$ denote redistribution according to Mehran’s (1976) class of linear inequality indices:

$$R = \int_0^1 \left[ L^*(p) - L_X(p) \right] k(p) \, dp$$

(1)

where $k(p)$ is an arbitrary positive function\(^2\).

Another well-known result is that a progressive tax which does not rerank individuals necessarily causes $L_T(p)$ to be dominated by $L_X(p)$, and $L_M(p)$ and $L^*(p)$ to dominate $L_X(p)$ (Jakobsson (1976)). Measurements of progressivity have thus naturally been based both on the distance between $L_X(p)$ and $L_T(p)$ and on the distance between $L_M(p)$ and $L_X(p)$, yielding indices based on what are called the tax redistribution (TR) and income redistribution (IR) views, respectively (see Pfähler (1987)). A tax $T$ is TR progressive if and only if $L_X$ dominates $L_T$, and a tax $T$ is IR progressive if and only if $L_M$ dominates $L_X$\(^3\). The farther is $L_T$ from $L_X$, or $L_X$ from $L_M$, the more TR or IR progressive is a tax\(^4\).

\(^2\) The indices are linear in the level of incomes. To see this, integrate by parts equation (1) by differentiating $L^*(p)$ and $L_X(p)$ with respect to $p$; this yields $R$ as the integral of a function linear in $X(p)$ and $M(p)$. The same linear class is used, for instance, to generate solutions to cooperative games in Blackorby, Bossert and Donaldson (1994). It is easily seen that inequality under $X$ will be greater than under $M$ for all indices satisfying $P$ if and only if $R$ is positive for all positive functions $k(p)$.

\(^3\) A benefit $B$ is, however, TR progressive if and only if its concentration curve $L_B$ dominates $L_X$.

\(^4\) There is some debate as to which of the IR and TR views is more appropriate as a basis for the measurement of progressivity (for an overview of this debate, see, for instance, Lambert (1993), ch.7). In the absence of reranking and when comparing two taxes yielding the same average tax rate, the two views are equivalent and yield the same ordering (Formby et al. (1990)). In general, however, the IR view is more closely linked to the income redistribution effected by a tax.
Testing whether a tax \( T_2 \) is more TR progressive than a tax \( T_1 \), we would then need to check whether \( L_{T_1} \) dominates \( L_{T_2} \); similarly, testing whether \( T_2 \) is more IR progressive than \( T_1 \) involves inferring whether \( L_{M_2} \) dominates \( L_{M_1} \), where \( M_i = X - T_i, \ i = 1, 2 \).

Analogously to (1), general classes of linear aggregate progressivity measures could be defined as

\[
AI R = \int_0^1 \left[ L_M(p) - L_X(p) \right] k(p) \, dp \\
AT R = \int_0^1 \left[ L_X(p) - L_T(p) \right] k(p) \, dp
\]

for positive weights \( k(p) \) (see Pfähler (1987)).

Plotnick (1982) posits three axioms for the measurement of horizontal inequity, which generally requires that “a tax should not alter the ordering of individuals by utility level” (see Feldstein (1976), p.83). These axioms are:

a) independence from the mean welfare level;

b) anonymity; and

c) a principle of inequity comparisons by which horizontal inequity is increased by permutations which move units farther from their equitable (e.g., initial) ranks.

When income is used as a measure of welfare, these axioms are consistent with the use of the distance between \( L_M(p) \) and \( L_T(p) \) as an indication of the presence of reranking and of horizontal inequity. It is well known, for instance, that \( L_M(p) \geq L_T(p) \) for all \( p \), with strict inequality somewhere, if and only if there is reranking of units in the redistributive process of moving from \( X \) to \( M \). The greater the distance between \( L_M \) and \( L_T \), the greater the extent of reranking. Before reranking can be considered as an appropriate measure of horizontal inequity, it is of course necessary that the initial ranking should be considered equitable. Thus we would not wish to use these indices to measure the impact of a policy which deliberately reranked units in response to a perceived initial inequity. With this proviso, we may introduce the following linear class of horizontal inequity indices (see Duclos (1993)):

\[
D = \int_0^1 \left[ L_M(p) - L_T(p) \right] k(p) \, dp
\]

(2)
We can also show that:

\[ R = AIR - D = \left( \frac{g}{1 - g} \right) AT R - D \]

where \( g \) is the average tax rate.

Special forms for \( k(p) \) have often been assumed. For \( k(p) = 2 \), for instance, we find with \( R \) the change in the Gini coefficient induced by taxation (the Musgrave-Thin measure of tax progressivity), with \( AIR \) the Reynolds-Smolensky index of vertical equity, with \( ATR \) the Kakwani index of tax redistribution (or tax departure from proportionality), and with \( D \) the Atkinson-Plotnick index of horizontal inequity. Using \( k(p) = X(p)/\mu_X \) for \( ATR \) yields the Suits index of tax redistribution.

In order to perform statistical inference on the incidence, in the form of redistribution, progressivity, and horizontal inequity, of taxes and benefits, we will establish in the next section the joint sampling distribution of \( L_X, L_M, L_T, \) and \( L^* \). This will enable us to

a) test whether \( L_X \) dominates \( L_T \), or whether \( L_M \) dominates \( L_X \), to determine whether \( T(X) \) is TR or IR progressive;

b) test whether \( L_T_1 \) is dominated by \( L_T_2 \), or whether \( L_M \) dominates \( L_M_2 \), to infer whether \( T_1 \) is more TR or IR progressive than \( T_2 \);

c) test the distance between \( L_M \) and \( L^* \) to assess the presence of reranking and horizontal inequity;

d) determine whether \( T \) is redistributive and inequality reducing by testing whether \( L^*(p) \) dominates \( L_X(p) \).

By weighting the joint sampling distribution of \( L_X, L_M, L_T \) and \( L^* \) with the weights \( k(p) \), we can derive the sampling distribution of any of the indices \( R, AIR, ATR, \) and \( D \). When performing inequality or welfare comparisons across time using panel data, we may think of \( L^*(p) \) and \( L_X(p) \) as two dependent Lorenz curves. Quantile-based poverty comparisons and indices can be seen as special cases of the above when the focus is put upon the lower portions of the income distributions (see Atkinson (1987) and Howes (1998)).
3. Asymptotic Distribution of Quantile-Based Estimators

Consider two jointly distributed random variables $Y$ and $Z$, and let $F$ denote the marginal distribution of $Z$. We are interested in estimating expectations of $Y$ conditional on $Z$ being in the low $p$-quantile of its distribution, that is, expectations like $\gamma_p \equiv E(Y \mid F(Z) \leq p)$. Formally:

$$p\gamma_p \equiv pE(Y \mid Z \leq G(p)) = E(Y \mid G_0, G([p]))(Z),$$

where $G$ is the inverse of $F$. Here the indicator function satisfies

$$I_{[0,0]}(Y) = \begin{cases} 1 & \text{if } Y \in [0, y]; \\ 0 & \text{otherwise}. \end{cases}$$

If $Y \equiv Z$, we note that $p\gamma_p$ yields the generalised Lorenz curve of $Y$ (Shorrocks (1983)), for which Beach and Davidson (1983) first derived the asymptotic sampling distribution.

Consider also a second set of two jointly distributed random variables $V$ and $W$, and let $F^*$ denote the marginal distribution of $W$. Then the conditional expectation $\delta_p$ of $V$, given that $W$ is below some $p$-quantile of its distribution, is defined by

$$p\delta_p \equiv pE(V \mid W \leq G^*(p)) = E(V \mid G_0, G^*[p])(W),$$

where $G^*$ is the inverse of $F^*$.

Next we define the following measure of distance between the expectations $\gamma_p$ and $\delta_p$:

$$\Gamma_p = p\left(\frac{\gamma_p}{\gamma_1} - \frac{\delta_p}{\delta_1}\right).$$

Depending on whether we are interested in the measurement of progressivity, horizontal equity, or redistribution, $\gamma_p$ and $\delta_p$ in (5) are chosen to take the following forms:

IR Progressivity:

$$\gamma_p = E(M \mid X \leq G(p)) \quad \delta_p = E(X \mid X \leq G(p));$$

Comparisons of IR Progressivity:

$$\gamma_p = E(M_1 \mid X \leq G(p)) \quad \delta_p = E(M_2 \mid X \leq G(p));$$

TR Progressivity:

$$\gamma_p = E(X \mid X \leq G(p)) \quad \delta_p = E(T \mid X \leq G(p));$$
Comparisons of TR Progressivity:

\[ \gamma_p = E(T_2 \mid X \leq G(p)) \quad \delta_p = E(T_1 \mid X \leq G(p)) \]

Horizontal Inequity:

\[ \gamma_p = E(M \mid X \leq G(p)) \quad \delta_p = E(M \mid M \leq G*(p)) \]

Redistribution:

\[ \gamma_p = E(M \mid M \leq G*(p)) \quad \delta_p = E(X \mid X \leq G(p)) \]

Suppose that \( N \) independent drawings have been made from the joint distribution of \( Y \) and \( Z \); write them as \((Y_i, Z_i), i = 1, \ldots, N\). An obvious estimator of \( p\gamma_p \) is then given by

\[
p\hat{\gamma}_p = N^{-1} \sum_{i=1}^{N} Y_i I_{[\hat{\gamma}_p]}(Z_i) \tag{6}
\]

where \( \hat{G}(p) \) is the sample estimate of the \( p \)-quantile of \( Z \). Similarly, we may estimate \( p\delta_p \) from a sample of \( N \) independent drawings \((V_i, W_i)\) by

\[
p\hat{\delta}_p = N^{-1} \sum_{i=1}^{N} V_i I_{[\hat{\delta}_p]}(W_i). \tag{7}
\]

Consider a set of \( K \) probabilities, \( p_i, i = 1, \ldots, K \), such that

\[ 0 < p_1 < p_2 < \ldots < p_K < 1. \]

Then let the \( K \)-vector \( \mathbf{P} \) be given by \( \mathbf{P} \equiv \{p_1, \ldots, p_K\}^{\top} \). The obvious estimator of \( \mathbf{P} \) is

\[
\hat{\mathbf{P}} = \left[ p_1 \left( \frac{\hat{\gamma}_{p_1}}{\hat{\gamma}_1} - \frac{\hat{\delta}_{p_1}}{\delta_1} \right) \ldots p_K \left( \frac{\hat{\gamma}_{p_K}}{\hat{\gamma}_1} - \frac{\hat{\delta}_{p_K}}{\delta_1} \right) \right]^{\top}, \tag{8}
\]

whose distribution naturally depends on the joint distribution of

\[
\hat{\Theta} \equiv \{p_1 \hat{\gamma}_{p_1}, \ldots, p_K \hat{\gamma}_{p_K}, \hat{\gamma}_1, p_1 \hat{\delta}_{p_1}, \ldots, p_K \hat{\delta}_{p_K}, \hat{\delta}_1\}^{\top}. \tag{9}
\]

Note that \( \hat{\gamma}_1 \) and \( \hat{\delta}_1 \) are just the sample means of \( Y \) and \( V \) respectively. Defining \( \Theta \) as the vector of the true values of the \( \gamma_p \) and the \( \delta_p \), as follows:

\[
\Theta = \left[ p_1 \gamma_{p_1}, \ldots, p_K \gamma_{p_K}, \gamma_1, p_1 \delta_{p_1}, \ldots, p_K \delta_{p_K}, \delta_1 \right]^{\top}
\]

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we will show that \( N^{-1/2}(\hat{\theta} - \theta) \) is asymptotically normal, with mean zero, and an asymptotic covariance matrix that can be estimated consistently without knowledge of the population distribution from which our sample was drawn. Our first result gives us the asymptotic covariance between \( p\hat{\gamma}_p \) and \( p'\hat{\delta}_{p'} \), for arbitrary \( 0 \leq p \leq 1 \) and \( 0 \leq p' \leq 1 \).

**Theorem 1:** Let the population second moments of \( Y \) and \( V \) conditional on \( Z \) and \( W \) be finite, and let the first moments be continuously differentiable in \( Z \) and \( W \). Further, let the marginal cumulative distribution functions of \( Z \) and \( W \) be strictly monotonic and continuously differentiable. Then the asymptotic covariance of \( p\hat{\gamma}_p \), as defined by (6), and \( p'\hat{\delta}_{p'} \), as defined by (7), is given by

\[
\lim_{N \to \infty} N \text{cov}(p\hat{\gamma}_p, p'\hat{\delta}_{p'}) = E(YV I_{[\hat{p}, \hat{G}(p)]}(Z) I_{[\hat{p}', \hat{G}^*(p')]}(W)) \\
- E(Y|Z = \hat{G}(p)) E(V|I_{[\hat{p}, \hat{G}(p)]}(Z) I_{[\hat{p}', \hat{G}^*(p')]}(W)) \\
- E(V|W = \hat{G}^*(p')) E(Y|I_{[\hat{p}, \hat{G}(p)]}(Z) I_{[\hat{p}', \hat{G}^*(p')]}(W)) \\
+ E(Y|Z = \hat{G}(p)) E(V|W = \hat{G}^*(p')) E(I_{[\hat{p}, \hat{G}(p)]}(Z) I_{[\hat{p}', \hat{G}^*(p')]}(W)) \\
- pp' \left( \hat{\gamma}_p - E(Y|Z = \hat{G}(p)) \right) \left( \hat{\delta}_{p'} - E(V|W = \hat{G}^*(p')) \right).
\]

**Proof:** See Appendix. \( \square \)

**Remarks and Corollaries:**

1. Everything in (10) can be estimated consistently in a distribution-free manner: \( \gamma_p \) and \( \delta_{p'} \) by \( \hat{\gamma}_p \) and \( \hat{\delta}_{p'} \), \( \hat{G}(p) \) and \( G^*(p') \) by \( \hat{G}(p) \) and \( G^*(p') \), that is, the sample \( p \) and \( p' \) quantiles of \( Z \) and \( W \) respectively. The unconditional expectations are readily estimated by their sample equivalents; thus for \( E(YV I_{[\hat{p}, \hat{G}(p)]}(Z) I_{[\hat{p}', \hat{G}^*(p')]}(W)) \), for instance, we may use the estimate

\[
N^{-1} \sum_{i=1}^{N} Y_i V_i I_{[\hat{p}, \hat{G}(p)]}(Z_i) I_{[\hat{p}', \hat{G}^*(p')]}(W_i),
\]

in which the sum is effectively over only those drawings \( i \) for which \( Z_i \) is less than or equal to the \( p \)-quantile of the sample distribution of \( Z \), and \( W_i \) is less than or equal to the \( p' \)-quantile of the sample distribution of \( W \). The conditional expectations are a little less obvious, but some form of kernel estimation can be used under weak regularity conditions.

2. The expression (10) gives explicitly only the covariances between elements of the vector \( \hat{\gamma} \equiv [\hat{\gamma}_p, \ldots, \hat{\gamma}_p] \) and those of \( \hat{\delta} \equiv [\hat{\delta}_{p'}, \ldots, \hat{\delta}_{p'}] \).

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However, in the covariance matrix of $\hat{\Theta}$, the diagonal block consisting of the variances and covariances of the elements of $\gamma$ can be obtained from (10) directly, by replacing $\delta_p$, $V$, $W$, and $G^*$ by $\gamma_p'$, $Y$, $Z$, and $G$ respectively. This procedure leads to certain simplifications, as seen in the following Corollary.

**Corollary 1** Under the conditions of Theorem 1, the asymptotic covariance between $\gamma_p$ and $\gamma_p'$, for $p \leq p'$, is given by

$$
\lim_{N \to \infty} N \text{ cov} \left( p\gamma_p, p'\gamma_p' \right) = p \left( \phi_p - \gamma_p^2 \right) + (1 - p') \left( E(Y \mid Z = G(p)) - \gamma_p \right) \left( E(Y \mid Z = G(p')) - \gamma_p' \right) + \left( E(Y \mid Z = G(p)) - \gamma_p \right) \left( \gamma_p' - \gamma_p \right),
$$

where

$$
p\phi_p \equiv E(Y^2 I_{[0, G(p)]}(Z)).
$$

If $p > p'$, the asymptotic covariance is obtained from (11) by inverting the roles of $p$ and $p'$.

**Proof:** Making the replacements given above in (10), noting that, for $p \leq p'$,

$$I_{[0, G(p')]}(Z) I_{[0, G(p)]}(Z) = I_{[0, G(p)]}(Z),
$$

and making use of the definition (12) yields the following expression for the asymptotic covariance of $p\gamma_p$ and $p'\gamma_p'$:

$$p \left( \phi_p - \gamma_p \right) E(Y \mid Z = G(p)) - \gamma_p \left( E(Y \mid Z = G(p')) \right) + E(Y \mid Z = G(p)) \left( E(Y \mid Z = G(p')) \right) - p' \left( \gamma_p - E(Y \mid Z = G(p')) \right) \left( \gamma_p' - E(Y \mid Z = G(p')) \right),$$

where can readily be seen to be equal to the more convenient form given in (11). The last statement in the enunciation of the Corollary follows from the symmetry of the covariance matrix of $\gamma$. 

(3) It is immediately clear that the asymptotic covariance matrix of $\hat{\delta}$ has exactly the same form as (11), with $\gamma_p$ and $\gamma_p'$ replaced by $\delta_p$ and $\delta_p'$, $Y$ and $Z$ replaced by $V$ and $W$, $G$ replaced by $G^*$, and $\phi_p$ replaced by...
\( \psi_p \equiv E(V^2 I_{[0,G^*(p)]}(W)) \). If we denote the full asymptotic covariance matrix of \( \hat{\Theta} \) by \( \Omega \), then the elements of the two off-diagonal blocks of \( \Omega \) are given by (10), and the elements of the diagonal blocks are given by (11) and its equivalent for \( \hat{\Delta} \).

(4) If \( Z \) and \( W \) are the same variable, as for instance in the cases listed above of IR and TR progressivity and comparison of progressivity, then the off-diagonal blocks of the asymptotic covariance matrix of \( \hat{\Theta} \) simplify in the same way as the diagonal blocks in (11).

**Corollary 2** Under the conditions of Theorem 1 with \( W \) the same variable as \( Z \), the asymptotic covariance between \( \hat{\gamma}_p \) and \( \hat{\delta}_p' \), for \( p \leq p' \), is given by

\[
\lim_{N \to \infty} \frac{1}{N} \text{cov} \left( p\hat{\gamma}_p, p'\hat{\delta}_p' \right) = \frac{1}{p} \left\{ \frac{1}{p} \left[ \left( 1 - p' \right) \left( \frac{1}{p} \left( E(Y|Z = G(p)) - \gamma_p \right) \right) \left( E(V|Z = G(p')) - \delta_p' \right) \right] \right\},
\]

where

\[
p\chi_p \equiv E(Y V I_{[0,G(p)]}(Z)).
\]

If \( p > p' \), the asymptotic covariance is obtained by exchanging the roles of \( p \), \( \gamma \), and \( Y \) with those of \( p' \), \( \delta \), and \( V \).

**Proof:** Exactly as for Corollary 1.

(5) In some cases, the variable \( Y \) may be the same as \( Z \), or may be a deterministic function of \( Z \). Or \( V \) may be a deterministic function of \( W \). In such cases, one or more of the conditional expectations in expressions (10), (11), or (13) become trivial to evaluate. For instance, in considering IR progressivity, \( \delta_p = E(X|X \leq G(p)) \). In our general notation, the variables \( Z \), \( W \), and \( V \) are all equal to \( X \). Thus we have, for instance,

\[
E(V|Z = G(p)) = E(X|X = G(p)) = G(p),
\]

and this quantity can be directly estimated as \( \bar{G}(p) \). The results of Beach and Davidson (1983), and of Beach, Davidson, and Slotsve (1994) then follow as special cases of the results presented here.
(6) One may legitimately wonder to what extent the results of the above Theorems and Corollaries are affected by the presence of measurement error. It is, after all, perfectly reasonable to suppose that things like incomes, taxes, and transfers are incorrectly reported in the available data sets, whether deliberately or otherwise. While intuition suggests that unbiased measurement errors will not seriously affect the results of this paper, the matter remains open, and merits further study.

Let us now return to the asymptotic covariance matrix of the estimator \( \hat{\theta} \) given in (8). Define the \( K \times 2(K+1) \) Jacobian \( J \) of the \( K \)-vector \( \Gamma \) as follows:

\[
J \equiv \begin{bmatrix}
\frac{\partial \Gamma_i}{\partial \theta_j}
\end{bmatrix} = \begin{bmatrix}
S(\gamma) & -S(\delta)
\end{bmatrix}
\]

where the \( K \times (K+1) \) matrices \( S(\gamma) \) and \( S(\delta) \) are given generically by the formula

\[
S(\alpha) = \begin{bmatrix}
1/\alpha_1 & \cdots & -\frac{p_1 \alpha_{p1}}{\alpha_1^2} \\
\vdots & \ddots & \vdots \\
1/\alpha_1 & \cdots & -\frac{p_K \alpha_{pK}}{\alpha_1^2}
\end{bmatrix}
\]

We can use a standard result of Rao (1973, pp.388-9) to state that:

**Theorem 2:** \( N^{1/2}(\hat{\theta} - \theta) \), as given by (8), has a \( K \)-variate normal limiting distribution with mean zero and covariance matrix \( J \Omega J^\top \).

Here \( \Omega \) is the asymptotic covariance matrix of the vector \( \theta \) given in (9).

**Proof:** Standard. \( \blacksquare \)

We have seen that all elements of \( J \Omega J^\top \) can be estimated consistently in a distribution-free manner, that is, without specifying an \textit{a priori} functional form for the population distribution. We can then use the results of Theorem 2 to perform statistical inference on population progressivity, horizontal inequity, and the amount of redistribution effected by various taxes and benefits. We illustrate this in the following section.
4. Illustration: the Canadian Tax and Benefit System

We use micro-data from the 1981 and 1990 Canadian Surveys of Consumer Finances. The distribution of income in Canada was subjected to a number of shocks between these two years, a feature shared with many other countries. The Canadian fiscal system was also significantly altered in that decade, particularly with the 1987 reform of personal income taxation which broadened the tax base, reduced the number of tax brackets, decreased the top marginal tax rates, converted a number of tax exemptions into tax credits, and aimed officially to simplify the collection of tax revenues and make it more “equitable”. A large number of changes were also made to important elements of the “safety net” programmes, such as the social assistance and unemployment insurance schemes.

The 1981 and 1990 Surveys contain, respectively, 37,779 and 45,461 observations on the distributions of incomes, income taxes, and a number of cash transfers. Families with negative gross or net incomes were removed. We use these data to compute gross incomes and the levels of taxes and transfers in the following four categories:

(a) Personal Income taxes (16.3%, 21.4%);
(b) Family and Youth Allowances, Child Tax credits and other tax credits and incomes from government sources (2.2%, 2.9%);
(c) Old Age transfers, including Old Age Security pensions, the Canada/-Quebec Pension Plan benefits, and the federal Guaranteed Income Supplement (7.6%, 11.4%); and
(d) Social Assistance and Unemployment Insurance benefits5 (3.6%, 5.7%).

The figures in parentheses indicate the percentages of gross incomes to which the relevant class of tax or transfers amount in 1981 and 1990 respectively. We use the OECD equivalence scale to adjust for varying household composition and size by transforming all monetary variables into “equivalised” ones. The conditional expectations of the variables $Y$ and $V$ in (10) were estimated using robust Gaussian kernel estimation (Silverman (1986), p.45). To test whether curve A dominates curve B, we reject the null hypothesis of non-dominance in favour of the alternative hypothesis of dominance only if each point of curve A is found to be statistically greater than the corresponding point on curve B at a 5% level of significance. This

5 Detailed information on the transfers in (b), (c) and (d) can be found, for instance, in Health and Welfare Canada (1992).
procedure, defended by Howes (1993), ensures that the probability of type I errors is never greater than 5%.

Figure 1 shows the IR progressivity of the personal income tax, measured as the distance between \( L_M(p) \) and \( L_X(p) \). The graphs show the differences for each decile, with error bars of twice the estimated asymptotic standard error (not always visible if the estimated standard error is small enough). The two plots show the measures of IR progressivity for both 1981 and 1990. Since the 1981 and 1990 samples are independently distributed, the asymptotic standard errors of the differences between the IR progressivity ordinates at each decile are straightforward to calculate from the standard errors of each ordinate. It is clear from the figure that IR progressivity was significantly greater in 1990 than in 1981 after the second decile. In Tables 1 and 2, the numerical values of the ordinates, along with their standard errors, are given, for 1990 and 1981 respectively. Although these tables are generally less simple to interpret than the graphs, they reveal that, for the first decile, the income tax was more IR progressive in 1981, and that the difference between the two years for the second decile is not significant.

In Figure 2 are shown the measures of TR progressivity of the personal income tax, that is, the difference between \( L_X(p) \) and \( L_T(p) \); Table 3 gives the numerical values for 1990. Over much of the range, it is clear that the 1981 personal income tax was more TR progressive than that of 1990, in contrast with the IR progressivity measures. Note also that the numerical values of this measure are much larger than those of Figure 1. This is simply due to the fact that \( L_M(p) \) and \( L_X(p) \), which are both concentration curves of incomes, are much closer than are \( L_X(p) \) and \( L_T(p) \), and it in no way suggests that TR progressivity is a more sensitive or sensible measure than IR progressivity. On the contrary, we feel that, for our present purposes at least, IR progressivity measures are more helpful than TR progressivity measures, and for the rest of this study, we concentrate on the former.

Tables 1, 2, and 3 also give measures of progressivity for the other groups of taxes and transfers, b), c), and d), mentioned above, as well as for the combination of all groups of taxes and benefits. All these other groups are found to be significantly more IR progressive in 1990 than in 1981. Hence, any member of the aggregate IR progressivity measures (AIR) would necessarily rank all of the groups of taxes and benefits, except the income tax, as more IR progressive in 1990 than in 1981. This can be seen graphically in Figure 3, which displays the same plots as in Figure 1,
but this time for the combination of all taxes and benefits. One sees clearly from Figures 1 and 3 that, although the incidence of the income tax is most progressive in the higher deciles, the combination of taxes and benefits is most progressive near the middle of the income distribution.

Direct comparison of the numbers in Tables 1 and 2, or of the error bars in Figures 1 and 3, enables us to determine which of the 1981 and 1990 tax and transfer systems is more IR progressive, that is, which one tends to redistribute the larger proportion of total income from the richer to the poorer portion of the population. However, if we wish to compare the different groups of taxes and transfers among themselves for one year, the comparison is complicated by the fact that the different ordinates, being estimated from the same sample, are therefore not independent. The comparisons, with appropriate standard errors that reflect this fact, are done numerically in Table 4 and graphically in Figure 4. It is perhaps most interesting to compare the other groups with the income tax, and it is these comparisons that are presented graphically in Figure 4. It is clearly not possible to declare Income Taxes statistically more or less IR progressive than any one of the three groups of benefits (Family and Youth Benefits, Old Age Transfers, and Social Assistance and Unemployment Insurance Benefits). At the lower deciles, Income Taxes are typically less IR progressive, but the ranking changes between the fourth and the seventh deciles. We also find that Family and Youth Benefits are significantly less IR progressive than either Old Age Transfers or Social Assistance and UI Benefits. This is mostly explained by the relatively small average size of the Family and Youth Benefits. The last column of Table 4 also shows that Old Age Transfers are more IR progressive than Social Assistance and UI Benefits. Exactly the same IR progressivity rankings apply for the year 1981, although the detailed results are not shown here for that year.

Tables 5 and 6 provide the Lorenz curve ordinates for the distribution of gross and net incomes, in 1990 and 1981 respectively, and also the ordinates of the concentration curve of net incomes when these net incomes are ordered in increasing values of gross incomes. We also tabulate the distance between the two Lorenz curves, which measures the extent of the redistribution effected by the tax and benefit system, and the distance between the concentration curve and the Lorenz curve for net incomes, which measures the degree of reranking of after-tax incomes relative to pre-tax incomes, or “horizontal inequity”, exerted by the system. This is also shown in Figure 5. Figure 6 displays the information concerning redistribution, that is, the distances between the Lorenz curves for gross and net incomes.
The redistributive impact of the tax and benefit system is visibly highly significant: the inequality of net incomes is unambiguously lower than that of gross incomes. Redistribution is at its highest around the fifth decile, at which point 8.3% of total income is transferred in 1990 from the richer (than the fifth decile) to the poorer part of the population. Note also that the tax and benefit system raises almost eightfold the total income share of the poorest 10% of the population. The extent of reranking is also precisely estimated, and reaches its maximum at the first decile. Whether this reranking should solely be regarded as horizontal inequity or rather the result of deliberate policies that aim to take more into account than mere “equivalised” incomes is of course not at all clear.

Comparisons of the ordinates from Tables 5 and 6, which contain independent statistics, can once again be made directly on the basis either of the reported standard errors or of the error bars in the Figures. We find that the distribution of gross incomes in 1981 is unambiguously and significantly more equal than in 1990. The distribution of net incomes in 1981 is, however, almost unambiguously more unequal than the distribution of net incomes in 1990, the only ambiguity being at the ninth decile, where the two curves cannot statistically be distinguished. Thus, in spite of the aggravation of the inequality of gross incomes in the 80’s, the 1990 tax and transfer system almost succeeds in making net incomes in 1990 unambiguously more equal than in 1981. It is therefore not surprising to find that the redistributive change in the inequality of incomes effected by the 1990 system is significantly greater than the change achieved by the 1981 system. At each of the deciles between the fourth and the eighth, for instance, 2% more of total income is redistributed from richer to poorer under the 1990 system than under the 1981 system, with a standard error of around 0.1%. Finally, it can easily be checked that horizontal inequity in 1990 is significantly greater than in 1981 at all deciles but the ninth, where the figures in Tables 5 and 6 are the same.

5. Conclusion

We have established the asymptotic sampling distribution of quantile-based estimators computed from samples which need not be independent. The results are particularly useful for the measurement of progressivity, redistribution and horizontal inequity, and for the measurement and comparisons of inequality, welfare, or poverty that make use of estimated quan-
tiles from possibly dependent samples. They also generalise most of the previous statistical inference results for the measurement of inequality and social welfare, and provide the statistical basis for the use of a number of popular indices.

Our illustrative application using the Canadian tax and benefit system shows that all groups of taxes and benefits are progressive, and that the elements of the 1990 system are generally more progressive than those in 1981. Among the groups of taxes and transfers, Old Age transfers are unambiguously the most progressive, followed by the Social Assistance and Unemployment Insurance benefits. Gross incomes are more equal in 1981 than in 1990, but net incomes are generally more equal in 1990 than in 1981. This is consistent with the finding that redistribution is significantly greater in 1990 than in 1981, with an associated increase in the extent of reranking, particularly at lower incomes.
Appendix

Proof of Theorem 1: Let the joint cumulative distribution function of $Z$ and $Y$ be denoted as $H(z,y)$, that is:

$$H(z,y) = \Pr(Z \leq z \text{ and } Y \leq y).$$

As in the text, $F$ denotes the c.d.f. of $Z$, supposed to be strictly monotonic and continuously differentiable, with inverse $G$. The quantity $\gamma_p$ of (3) can then be characterised by

$$p \gamma_p = \int_0^{G(p)} \int_0^\infty y \, d^2 H(z,y),$$

where the integral from 0 to $G(p)$ applies to $z$, and that from 0 to $\infty$ to $y$. The estimate $\hat{\gamma}_p$ of (6) is similarly given by

$$p \hat{\gamma}_p = \int_0^{\hat{G}(p)} \int_0^\infty y \, d^2 \hat{H}(z,y),$$

(14)

where $\hat{G}$ is defined as before as the sample quantile function for $Z$, and $\hat{H}$ is the empirical distribution of $Z$ and $Y$ jointly.

The integral in (14) can be split up into an integral (over $z$) from 0 to $G(p)$ and another from $G(p)$ to $\hat{G}(p)$. The first is easy to deal with, as it can be written as a sum of i.i.d. variables:

$$\int_0^{G(p)} \int_0^\infty y \, d^2 \hat{H}(z,y) = N^{-1} \sum_{i=1}^N Y_i \ell_{[0,G(p)]}(Z_i).$$

(15)

The second integral is of order $N^{-1/2}$, and it can be approximated as follows:

$$\int_{G(p)}^{\hat{G}(p)} \int_0^\infty y \, d^2 \hat{H}(y,z) = \int_{G(p)}^{\hat{G}(p)} \int_0^\infty y \, d^2 \hat{H}(y,z) + O(N^{-1})$$

$$= \int_{G(p)}^{\hat{G}(p)} E(Y | z) \, dF(z) + O(N^{-1}).$$

This proof is based on that found in the Appendix of Beach, Davidson, and Slotsve (1994), but extends it considerably.
The first equality follows from the root-$N$ consistency of $\hat{H}$ for $H$, and the second from the definition of the conditional expectation $E(Y \mid z)$. Since we assume that this conditional expectation is a smooth function of $z$, we have for $z \in [G(p), \bar{G}(p)]$ that

$$E(Y \mid z) = E(Y \mid Z = G(p)) + O(N^{-1/2}).$$

Thus, approximately with error of order only $N^{-1}$, the second integral is

$$E(Y \mid Z = G(p)) \int_{G(p)}^{\bar{G}(p)} d\hat{F}(z),$$

(using the root-$N$ consistency of $\hat{F}$), which can be rewritten as

$$-E(Y \mid Z = G(p))(\hat{F}(G(p)) - p) = pE(Y \mid Z = G(p)) - \int_{G(p)}^{\bar{G}(p)} E(Y \mid Z = G(p)) d\hat{F}(z). \quad (16)$$

The expressions in (15) and (16) can be added together to obtain an asymptotic approximation for (14). To leading order,

$$p\tilde{\gamma}_p = pE(Y \mid Z = G(p)) + \int_{G(p)}^{\bar{G}(p)} \int_{0}^{\infty} \left( y - E(Y \mid Z = G(p)) \right) d^2 \hat{H}(z, y)$$

$$= pE(Y \mid Z = G(p)) + N^{-1} \sum_{i=1}^{N} \left( Y_i - E(Y \mid Z = G(p)) \right) I_{\|G(p)\|}(Z_i). \quad (17)$$

Similarly we have

$$p\tilde{\delta}_p = pE(V \mid W = G^*(p)) + \int_{0}^{G^*(p)} \int_{0}^{\infty} \left( v - E(V \mid W = G^*(p)) \right) d^2 \hat{L}(w, v)$$

$$= pE(V \mid W = G^*(p)) + N^{-1} \sum_{i=1}^{N} \left( V_i - E(V \mid W = G^*(p)) \right) I_{\|G^*(p)\|}(W_i). \quad (18)$$

where $\hat{L}(w, v)$ denotes the empirical joint c.d.f. of $W$ and $V$, and $G^*$ is the inverse of the c.d.f. of $W$. 

- 18 -
The expectation of $p \gamma_p$ can readily be calculated from (17). To leading order:

$$
E(p \gamma_p) = p E(Y | Z = G(p)) + E\left(Y - E(Y | Z = G(p)) \ I_{[0, G(p)]}(Z)\right)
$$

$$
= p E(Y | Z = G(p)) + p \gamma_p - p E(Y | Z = G(p))
$$

which demonstrates the consistency of the estimator. Similarly, the fact that both (17) and (18) are sums of independent identically distributed random variables with finite second moments leads immediately to their asymptotic normality by the central limit theorem.

The covariance structure can now be obtained by simple calculation, based once more on the structure of (17) and (18) as sums of i.i.d. variables. We have

$$
\lim_{N \to \infty} N \ \text{cov}(p \gamma_p, p \delta_p) =
$$

$$
E\left(\left(Y - E(Y | Z = G(p))\right) I_{[0, G(p)]}(Z) \left(V - E(V | W = G^*(p'))\right) I_{[0, G^*(p')]}(W)\right)
$$

$$
- E\left(Y - E(Y | Z = G(p)) \ I_{[0, G(p)]}(Z) \left(V - E(V | W = G^*(p'))\right) I_{[0, G^*(p')]}(W)\right)
$$

$$
E\left(V - E(V | W = G^*(p')) \ I_{[0, G^*(p')]}(W)\right).
$$

Now by the definitions of $\gamma_p$ and $\delta_p$, we have

$$
E\left(Y - E(Y | Z = G(p)) \ I_{[0, G(p)]}(Z)\right) = p \left(\gamma_p - E(Y | Z = G(p))\right)
$$

and

$$
E\left(V - E(V | W = G^*(p')) \ I_{[0, G^*(p')]}(W)\right) = p' \left(\delta_p - E(V | W = G^*(p'))\right).
$$

Thus the last term on the right-hand side of (19) equals the last term on the right-hand side of (10). The first term on the right-hand side of (19), when expanded, yields the other terms on the right-hand side of (10).
References


Figure 1
IR Progressivity of the Personal Income Tax

Figure 2
TR Progressivity of the Personal Income Tax
Figure 3
IR Progressivity of All Taxes and Benefits

Figure 4
IR Progressivity relative to Income Taxes (1990)
Figure 5
Measure of Reranking

Figure 6
Comparison of Lorenz Curves for Gross and Net Incomes
Table 1

IR Progressivity for the 1990 Tax and Benefit System

*asymptotic standard errors in italics*

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### Table 3

**TR Progressivity for the 1990 Tax and Benefit System**

*asymptotic standard errors in italics*

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# Table 4

**Comparisons of IR Progressivity for the 1990 System**

*asymptotic standard errors in italics*

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<th>Income Taxes Family and Youth Benefits</th>
<th>Income Taxes Old Age Transfers</th>
<th>Income Taxes Social Assistance and UI Benefits</th>
<th>Family and Youth Benefits Old Age Transfers</th>
<th>Family and Youth Benefits Social Assistance and UI Benefits</th>
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Table 5

**Horizontal Inequity and Redistribution for the 1990 System**

*asymptotic standard errors in italics*

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<th>Lorenz curve for gross incomes</th>
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Table 6

Horizontal Inequity and Redistribution for the 1981 System

*asymptotic standard errors in italics*

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<th>Lorenz curve for gross incomes</th>
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